NON-HAAR p-ADIC WAVELETS AND THEIR APPLICATION TO PSEUDO-DIFFERENTIAL OPERATORS AND EQUATIONS

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ABSTRACT. In this paper a countable family of new compactly supported *non-Haar* p-adic wavelet bases in $\mathcal{L}^2(\mathbb{Q}_p^n)$ is constructed. We use the wavelet bases in the following applications: in the theory of p-adic pseudo-differential operators and equations. Namely, we study the connections between wavelet analysis and spectral analysis of p-adic pseudo-differential operators. A criterion for a multidimensional p-adic wavelet to be an eigenfunction for a pseudo-differential operator is derived. We prove that these wavelets are eigenfunctions of the fractional operator. In addition, p-adic wavelets are used to construct solutions of linear and semi-linear pseudo-differential equations. Since many p-adic models use pseudo-differential operators (fractional operator), these results can be intensively used in these models.

1. INTRODUCTION

According to the well-known Ostrovsky theorem, there are two equal in rights "universes": the real "universe" and the p-adic one. The real "universe" is based on the field $\mathbb{R}$ of real numbers, which is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the usual Euclidean distance between rational numbers. In its turn, the p-adic "universe" is based on the field $\mathbb{Q}_p$ of p-adic numbers, which is defined as the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$. This norm is defined as follows: if an arbitrary rational number $x \neq 0$ is represented as $x = p^{\gamma}\frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers m, n are not divisible by p, then

$$|x|_p = p^{-\gamma}, \quad x \neq 0, \qquad |0|_p = 0. \tag{1.1}$$

The norm $|\cdot|_p$ satisfies the strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$ and is *non-Archimedean*.

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During a few hundred years theoretical physics has been developed on the basis of real (and later also complex) numbers. However, in the last 20 years the field of p-adic numbers $\mathbb{Q}_p$ (as well as its algebraic extensions) has been intensively used in theoretical and mathematical physics, stochastics, cognitive sciences and psychology [6], [7], [11]– [13], [16], [22], [26], [35]– [37] (see also the references therein). However, since a p-adics is a young area there are many unsufficiently studied problems which have been intensively studied in the real "universe". One of them is the p-adic wavelet theory.

Nowadays it is difficult to find an engineering area where wavelets (in the real setting) are not applied. There is a general scheme for the construction of wavelets in the real setting, which was developed in the early nineties. This scheme is based on the notion of the *multiresolution analysis* introduced by Y. Meyer and S. Mallat [29], [30]. The p-adic wavelet theory is now in conceptual stage of investigation. In this theory the situation is as follows.

In 2002, S. V. Kozyrev [24] found a compactly supported p-adic wavelet basis for $\mathcal{L}^2(\mathbb{Q}_p)$ which is an analog of the real Haar basis:

$$\theta_{k;ja}(x) = p^{-j/2}\chi_p\big(p^{-1}k(p^jx-a)\big)\Omega\big(|p^jx-a|_p\big), \quad x \in \mathbb{Q}_p, \tag{1.2}$$

$k=1,2,\dots,p-1$, $j \in \mathbb{Z}$, $a \in I_p = \mathbb{Q}_p/\mathbb{Z}_p$, where $\Omega(t)$ is the characteristic function of the segment $[0,1] \subset \mathbb{R}$, the function $\chi_p(\xi x)$ is an *additive character* of the field $\mathbb{Q}_p$ for every fixed $\xi \in \mathbb{Q}_p$ (see Sec. 2). Kozyrev's wavelet basis (1.2) is generated by dilatations and translations of the wavelet functions:

$$\theta_k(x) = \chi_p(p^{-1}kx)\Omega(x|_p), \quad x \in \mathbb{Q}_p, \quad k=1,2,\dots,p-1. \tag{1.3}$$

Multidimensional p-adic bases obtained by direct multiplying out the wavelets (1.2) were considered in [2]. The Haar wavelet basis (1.2) was extended to the ultrametric spaces in [14], [15], [25].

J. J. Benedetto and R. L. Benedetto [8], R. L. Benedetto [9] suggested a method for finding wavelet bases on the locally compact abelian groups with compact open subgroups, which includes the p-adic setting. They did not develop the *multiresolution analysis* (*MRA*), their method being based on the *theory of wavelet sets*. Their method only allows the construction of wavelet functions whose Fourier transforms are the characteristic functions of some sets (see [8, Proposition 5.1.]). Note that Kozyrev's wavelet basis (1.2) can be constructed in the framework of Benedettos' approach [8, 5.1.].

The notion of p-adic MRA was introduced and a general scheme for its construction was described in [32]. To construct a p-adic analog of a classical MRA we need a proper p-adic *refinement equation*. In our preprint [17], the following conjecture was proposed: the equality

$$\phi(x) = \sum_{r=0}^{p-1} \phi\Big(\frac{1}{p}x - \frac{r}{p}\Big), \quad x \in \mathbb{Q}_p, \tag{1.4}$$

can be considered as a *refinement equation*. A solution ϕ to this equation (*a refinable function*) is the characteristic function of the unit disc

$$\phi(x) = \Omega\big(|x|_p\big), \quad x \in \mathbb{Q}_p. \tag{1.5}$$

The equation (1.4) reflects *natural* "self-similarity" of the space $\mathbb{Q}_p$: the unit disc $B_0(0) = \{x : |x|_p \le 1\}$ is represented by a sum of p mutually *disjoint* discs

$$B_0(0) = B_{-1}(0) \cup \Big(\cup_{r=1}^{p-1} B_{-1}(r)\Big),$$

where $B_{-1}(r) = \big\{x : |x-r|_p \le p^{-1}\big\}$ (see formula (2.6) in Proposition 2.1). The equation (1.4) is an analog of the *refinement equation* generating the Haar MRA in the real analysis. Using this idea, the notion of p-adic MRA was introduced and a general scheme for its construction was described in [32]. The scheme was realized for construction 2-adic Haar MRA with using (1.5) as the *generating refinement equation*. In contrast to the real setting, the *refinable function* ϕ generating the Haar MRA is *periodic*, which *never holds* for real refinable functions. Due to this fact, there exist *infinity many different* orthonormal wavelet bases in the same Haar MRA (see [32]). One of them coincides with Kozyrev's wavelet basis (1.2). From the standpoint of results of the papers [19], [1], in [32] all compactly supported wavelet Haar bases were constructed.

It turned out that the above-mentioned p-adic wavelets are eigenfunctions of p-adic pseudo-differential operators [2]–[4], [17], [18], [24], [25] (see also Sec. 4). Thus the spectral theory of p-adic pseudo-differential operators is related to the wavelet theory. On the other hand, it is well-known that numerous models connected with p-adic differential equations use pseudo-differential operators (see [12], [22], [35] and the references therein). This is closely related to the fact that for the p-adic analysis associated with the mapping $\mathbb{Q}_p \to \mathbb{C}$, the operation of differentiation is *not defined*, and as a result, many models connected with p-adic differential equations use pseudo-differential operators, in particular, the fractional operator D^{α} (see the above-mentioned papers and books). These two facts imply that study of wavelets is important since it gives a new powerful technique for solving p-adic problems.

Contents of the paper. The main goal of this paper is to construct a *countable family of new compactly supported non-Haar p-adic wavelet bases* in $\mathcal{L}^2(\mathbb{Q}_p)$. Another goal is to study the connections between *wavelet analysis and spectral analysis of p-adic pseudo-differential operators*. In addition, we use our results to solve the Cauchy problems for p-adic pseudo-differential equations.

In Sec. 2, we recall some facts from the theory of p-adic distributions [10], [33]– [35]. In particular, in Subsec. 2.2, some facts from the theory of the p-adic Lizorkin spaces of test functions $\Phi(\mathbb{Q}_p^n)$ and distributions $\Phi'(\mathbb{Q}_p^n)$ are recalled (for details, see [2]).

In Sec. 3, *non-Haar p-adic compactly supported wavelet bases* are introduced. In Subsec. 3.1, we construct the non-Haar basis (3.4) which was introduced in the preprint [17] (for the brief review see [18]). In contrast to (1.2), for the basis

(3.4) the number of generating wavelet functions is not minimal, for example, for $p=2$ we have 2^{m-1} wavelet functions (instead of one as it is for (1.2) and for classical wavelet bases in real analysis). The basis (3.4) is the *non-Haar* wavelet basis, since it cannot be constructed in the framework of the p-adic Haar MRA (see [32] and Theorem 3.1). According to Remark 3.1, Kozyrev's wavelet basis (1.2) is a particular case of the basis (3.4) for $m=1$. According to the same remark, our non-Haar wavelet basis (3.4) can be obtained by using the algorithm developed by the Benedettos [8]. However, using our approach, we obtained the *explicit formulas* (3.4) for this basis. Moreover, our technique allows to produce new wavelet bases (see in Subsec. 3.2). In Subsec. 3.2, using the proof scheme of [32, Theorem 1], we construct *infinitely many new different non-Haar wavelet bases* (3.23), (3.17), (3.18) which are distinct from the basis (3.4). These new bases cannot be obtained in the framework of the standard scheme of the MRA [32]. Our bases given by formulas (3.23), (3.17), (3.18) *cannot be constructed* by Benedettos' method [8]. For example, it is easy to see that the Fourier transform of our generating wavelet-functions $\psi_s^{(m)[1]}(x)$, $s \in J_{p;m}$ defined by (3.17), (3.18) and all their shifts *are not characteristic functions* (see Remark 3.2). In Subsec. 3.3, n-dimensional non-Haar wavelet bases (3.25) and (3.29) are introduced as n-direct products of the corresponding one-dimensional non-Haar wavelet bases. All above wavelets belong to the Lizorkin space of test functions $\Phi(\mathbb{Q}_p^n)$. In Subsec. 3.4, the characterizations of the spaces of Lizorkin test functions and distributions in terms of wavelets are given (see Lemma 3.1 and Proposition 3.1), which are very useful for solution of p-adic pseudo-differential equations. The assertions of the type of Lemma 3.1 and Proposition 3.1 were stated for ultrametric Lizorkin spaces in [5].

In Sec. 4, the spectral theory of one class of p-adic multidimensional pseudo-differential operators (4.1) (which were introduced in [2]) is studied. In Subsec. 4.1, 4.2, we recall some facts on this class of pseudo-differential operators defined in the Lizorkin space $\mathcal{D}'(\mathbb{Q}_p^n)$. Our operators (4.1) include the fractional operator [33, §2], [34, III.4.] and the pseudo-differential operators studied in [22], [38], [39]. The Lizorkin spaces are *invariant* under our pseudo-differential operators. In Subsec. 4.3, by Theorems 4.1, 4.2 the criterion (4.6) for multidimensional p-adic pseudo-differential operators (4.1) to have multidimensional wavelets (3.25) and (3.29) as eigenfunctions is derived. In particular, the multidimensional wavelets (3.25) and (3.29) are eigenfunctions of the Taibleson fractional operator (see Corollaries 4.2–4.4).

In Sec. 5, the results of Sec. 3, 4 are used to solve the Cauchy problems for p-adic evolutionary pseudo-differential equations. Note that the Cauchy problem (5.16) was solved in [4] for a particular case. These results give significant advance in the theory of p-adic pseudo-differential equations. Moreover, since many p-adic models use pseudo-differential operators (in particular, fractional operator), these results can be used in applications.

It easy to see that formulas (3.23), (3.17), (3.18) do not give description of all *non-Haar wavelet bases*.

Due to the results of Sec. 3, there arise two important problems: to construct an analog of MRA scheme and describe all compactly supported *non-Haar wavelet bases*. It is necessary to verify if all *non-Haar wavelet bases* are given by formulas (3.23), (3.17), (3.18)?

Taking into account representation (3.15), it is natural to suggest that in this case we must use the *refinement type equation*:

$$\phi(x)=\sum_b \phi\Big(\frac{1}{p^m}x-\frac{b}{p^m}\Big), \quad x\in\mathbb{Q}_p,$$

instead of the *Haar refinement equation* (1.4), where $b=0$ or $b=b_rp^r+b_{r+1}p^{r+1}+\cdots+b_{m-1}p^{m-1}$, $r=0,1,\dots,m-1$, $0\le b_j\le p-1$, $b_r\ne 0$. This equation reflects the geometric fact that the unit disc $B_0=\{x:|x|_p\le 1\}$ is represented by a sum of p^m mutually *disjoint* discs $B_{-m}(b)=\{x:|x-b|_p\le p^{-m}\}$.

2. Preliminary results in p-adic analysis

2.1. **p-Adic functions and distributions.** We shall systematically use the notations and results from [35]. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the sets of positive integers, integers, complex numbers, respectively.

Any p-adic number $x\in\mathbb{Q}_P$, $x\ne 0$, is represented in the *canonical form*

$$x=p^\gamma(x_0+x_1p+x_2p^2+\cdots) \tag{2.1}$$

where $\gamma=\gamma(x)\in\mathbb{Z}$, $x_k=0,1,\dots,p-1$, $x_0\ne 0$, $k=0,1,\dots$. The series is convergent in the p-adic norm $|\cdot|_p$, and one has $|x|_p=p^{-\gamma}$. The *fractional part* of a number $x\in\mathbb{Q}_p$ (given by (2.1)) is defined as follows

$$\{x\}_p=\begin{cases}0, & \text{if} \quad \gamma(x)\ge 0 \quad \text{or} \quad x=0,\\ p^\gamma(x_0+x_1p+x_2p^2+\cdots+x_{|\gamma|-1}p^{|\gamma|-1}), & \text{if} \quad \gamma(x)<0.\end{cases} \tag{2.2}$$

The space $\mathbb{Q}_p^n=\mathbb{Q}_p\times\cdots\times\mathbb{Q}_p$ consists of points $x=(x_1,\dots,x_n)$, where $x_j\in\mathbb{Q}_p$, $j=1,2\dots,n$, $n\ge 2$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p=\max_{1\le j\le n}|x_j|_p, \quad x\in\mathbb{Q}_p^n, \tag{2.3}$$

where $|x_j|_p$, $x_j\in\mathbb{Q}_p$, is defined by (1.1), $j=1,\dots,n$. Denote by $B_\gamma^n(a)=\{x:|x-a|_p\le p^\gamma\}$ the ball of radius p^γ with the center at a point $a=(a_1,\dots,a_n)\in\mathbb{Q}_p^n$ and by $S_\gamma^n(a)=\{x:|x-a|_p=p^\gamma\}=B_\gamma^n(a)\setminus B_{\gamma-1}^n(a)$ its boundary (sphere), $\gamma\in\mathbb{Z}$. For $a=0$ we set $B_\gamma^n(0)=B_\gamma^n$ and $S_\gamma^n(0)=S_\gamma^n$. For the case $n=1$ we will omit the upper index n. Here

$$B_\gamma^n(a)=B_\gamma(a_1)\times\cdots\times B_\gamma(a_n), \tag{2.4}$$

where $B_\gamma(a_j)=\{x_j:|x_j-a_j|_p\le p^\gamma\}$ is a disc of radius p^γ with the center at a point $a_j\in\mathbb{Q}_p$, $j=1,2\dots,n$. Any two balls in $\mathbb{Q}_p^n$ either are disjoint or one contains the other. Every point of the ball is its center.

Proposition 2.1. ([35, I.3, Examples 1,2.]) *The disc* B_γ *is represented by the sum of* $p^{\gamma-\gamma'}$ disjoint *discs* $B_{\gamma'}(a)$, $\gamma'<\gamma$*:*

$$B_\gamma = B_{\gamma'} \cup \cup_a B_{\gamma'}(a), \tag{2.5}$$

where $a=0$ *and* $a=a_{-r}p^{-r}+a_{-r+1}p^{-r+1}+\cdots+a_{-\gamma'-1}p^{-\gamma'-1}$ *are the centers of the discs* $B_{\gamma'}(a)$, $0\le a_j\le p-1$, $j=-r,-r+1,\dots,-\gamma'-1$, $a_{-r}\ne 0$, $r=\gamma,\gamma-1,\gamma-2,\dots,\gamma'+1$. *In particular, the disc* B_0 *is represented by the sum of* p disjoint *discs*

$$B_0 = B_{-1} \cup \cup_{r=1}^{p-1} B_{-1}(r), \tag{2.6}$$

where $B_{-1}(r)=\{x\in S_0 : x_0=r\}=r+p\mathbb{Z}_p$, $r=1,\dots,p-1$*;* $B_{-1}=\{|x|_p\le p^{-1}\}=p\mathbb{Z}_p$*; and* $S_0=\{|x|_p=1\}=\cup_{r=1}^{p-1}B_{-1}(r)$. *Here all the discs are disjoint.*

We call covering (2.5), (2.6) the *canonical covering* of the disc B_0.

A complex-valued function f defined on $\mathbb{Q}_p^n$ is called *locally-constant* if for any $x\in\mathbb{Q}_p^n$ there exists an integer $l(x)\in\mathbb{Z}$ such that

$$f(x+y)=f(x), \quad y\in B^n_{l(x)}.$$

Let $\mathcal{E}(\mathbb{Q}_p^n)$ and $\mathcal{D}(\mathbb{Q}_p^n)$ be the linear spaces of locally-constant $\mathbb{C}$-valued functions on $\mathbb{Q}_p^n$ and locally-constant $\mathbb{C}$-valued functions with compact supports (so-called test functions), respectively; $\mathcal{D}(\mathbb{Q}_p)$, $\mathcal{E}(\mathbb{Q}_p)$ [35, VI.1.,2.]. If $\varphi\in\mathcal{D}(\mathbb{Q}_p^n)$, according to Lemma 1 from [35, VI.1.], there exists $l\in\mathbb{Z}$, such that

$$\varphi(x+y)=\varphi(x), \quad y\in B^n_l, \quad x\in\mathbb{Q}_p^n.$$

The largest of the numbers $l=l(\varphi)$ is called the *parameter of constancy* of the function φ. Let us denote by $\mathcal{D}^l_N(\mathbb{Q}_p^n)$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^n)$ with supports in the ball B^n_N and with parameters of constancy $\ge l$ [35, VI.2.]. We have $\mathcal{D}^l_N(\mathbb{Q}_p^n)\subset\mathcal{D}^{l'}_{N'}(\mathbb{Q}_p^n)$, $N\le N'$, $l\ge l'$.

Lemma 2.1. ([35, VI.5.,(5.2')]) *Any function* $\varphi\in\mathcal{D}^l_N(\mathbb{Q}_p^n)$ *can be represented as a finite linear combination*

$$\varphi(x)=\sum_{\nu=1}^{p^{n(N-l)}}\varphi(a^\nu)\Delta_l(x-a^\nu), \quad x\in\mathbb{Q}_p^n,$$

where $\Delta_l(x-a^\nu)=\Omega(p^{-l}|x-a^\nu|_p)](x)$ *is the characteristic function of the ball* $B^n_l(a^\nu)$*, and the points* $a^\nu=(a^\nu_1,\dots a^\nu_n)\in B^n_N$ *do not depend on* φ *and are such that the bolls* $B^n_l(a^\nu)$, $\nu=1,\dots,p^{n(N-l)}$*, are disjoint and cover the ball* B^n_N.

Denote by $\mathcal{D}'(\mathbb{Q}_p^n)$ the set of all linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ [35, VI.3.].

The Fourier transform of $\varphi\in\mathcal{D}(\mathbb{Q}_p^n)$ is defined by the formula

$$F[\varphi](\xi)=\int_{\mathbb{Q}_p^n}\chi_p(\xi\cdot x)\varphi(x)\,d^nx, \quad \xi\in\mathbb{Q}_p^n,$$

where $d^n x = dx_1 \cdots dx_n$ is the Haar measure such that $\int_{|\xi|_p \le 1} d^n x = 1$; $\chi_p(\xi \cdot x) = \chi_p(\xi_1 x_1) \cdots \chi_p(\xi_n x_n)$; $\xi \cdot x$ is the scalar product of vectors and $\chi_p(\xi_j x_j) = e^{2\pi i \{\xi_j x_j\}_p}$ are additive characters, $\{x\}_p$ is the *fractional part* (2.2) of a number $x \in \mathbb{Q}_p$.

Lemma 2.2. ([33, Lemma A.], [34, III,(3.2)], [35, VII.2.]) *The Fourier transform is a linear isomorphism $\mathcal{D}(\mathbb{Q}_p^n)$ into $\mathcal{D}(\mathbb{Q}_p^n)$. Moreover,*

$$\varphi(x) \in \mathcal{D}_N^l(\mathbb{Q}_p^n) \quad \textit{iff} \quad F\big[\varphi(x)\big](\xi) \in \mathcal{D}_{-l}^{-N}(\mathbb{Q}_p^n). \tag{2.7}$$

The Fourier transform $F[f]$ of a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ is defined by the relation $\langle F[f], \varphi \rangle = \langle f, F[\varphi] \rangle$, for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$.

Let A be a matrix and $b \in \mathbb{Q}_p^n$. Then for a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ the following relation holds [35, VII,(3.3)]:

$$F[f(Ax+b)](\xi) = |\det A|_p^{-1} \chi_p\big(-A^{-1}b \cdot \xi\big) F[f(x)]\big(A^{-1}\xi\big), \tag{2.8}$$

where $\det A \ne 0$. According to [35, IV,(3.1)],

$$F[\Omega(p^{-k}|\cdot|_p)](x) = p^{nk}\Omega(p^k|x|_p), \quad k \in \mathbb{Z}, \qquad x \in \mathbb{Q}_p^n. \tag{2.9}$$

In particular, $F[\Omega(|\xi|_p)](x) = \Omega(|x|_p)$. Here $\Omega(t)$ is the characteristic function of the segment $[0,1] \subset \mathbb{R}$,

2.2. **p-Adic Lizorkin spaces.** According to [2], [3], the p-adic *Lizorkin space of test functions* is defined as

$$\Phi(\mathbb{Q}_p^n) = \{\phi : \phi = F[\psi],\, \psi \in \Psi(\mathbb{Q}_p^n)\},$$

where $\Psi(\mathbb{Q}_p^n) = \{\psi(\xi) \in \mathcal{D}(\mathbb{Q}_p^n) : \psi(0) = 0\}$. The space $\Phi(\mathbb{Q}_p^n)$ can be equipped with the topology of the space $\mathcal{D}(\mathbb{Q}_p^n)$ which makes it a complete space. In view of Lemma 2.2, the Lizorkin space $\Phi(\mathbb{Q}_p^n)$ admits the following characterization.

Lemma 2.3. ([2], [3]) (a) $\phi \in \Phi(\mathbb{Q}_p^n)$ *iff* $\phi \in \mathcal{D}(\mathbb{Q}_p^n)$ *and*

$$\int_{\mathbb{Q}_p^n} \phi(x)\, d^n x = 0. \tag{2.10}$$

(b) $\phi \in \mathcal{D}_N^l(\mathbb{Q}_p^n) \cap \Phi(\mathbb{Q}_p^n)$, *i.e.,* $\int_{B_N^n} \phi(x)\, d^n x = 0$, *iff* $\psi = F^{-1}[\phi] \in \mathcal{D}_{-l}^{-N}(\mathbb{Q}_p^n) \cap \Psi(\mathbb{Q}_p^n)$, *i.e.,* $\psi(\xi) = 0$, $\xi \in B_{-N}^n$.

Suppose that $\Phi'(\mathbb{Q}_p^n)$ and $\Psi'(\mathbb{Q}_p^n)$ denote the topological dual of the spaces $\Phi(\mathbb{Q}_p^n)$ and $\Psi(\mathbb{Q}_p^n)$, respectively. We call $\Phi'(\mathbb{Q}_p^n)$ the space of p-adic *Lizorkin distributions*. The space $\Phi'(\mathbb{Q}_p^n)$ can be obtained from $\mathcal{D}'(\mathbb{Q}_p^n)$ by "sifting out" constants. Thus two distributions in $\mathcal{D}'(\mathbb{Q}_p^n)$ differing by a constant are indistinguishable as elements of $\Phi'(\mathbb{Q}_p^n)$.

We define the Fourier transform of $f \in \Phi'(\mathbb{Q}_p^n)$ and $g \in \Psi'(\mathbb{Q}_p^n)$ respectively by formulas $\langle F[f], \psi \rangle = \langle f, F[\psi] \rangle$, for all $\psi \in \Psi(\mathbb{Q}_p^n)$, and $\langle F[g], \phi \rangle = \langle g, F[\phi] \rangle$, for all $\phi \in \Phi(\mathbb{Q}_p^n)$. It is clear that $F[\Phi'(\mathbb{Q}_p^n)] = \Psi'(\mathbb{Q}_p^n)$ and $F[\Psi'(\mathbb{Q}_p^n)] = \Phi'(\mathbb{Q}_p^n)$ [2].

Recall that in the real setting the Lizorkin spaces were introduced in the excellent papers by P. I. Lizorkin [27], [28].

3. Non-Haar p-adic wavelet bases

3.1. **One non-Haar wavelet basis in $\mathcal{L}^2(\mathbb{Q}_p)$.** It is well known that $\mathbb{Q}_p = B_0(0)\cup\cup_{\gamma=1}^{\infty}S_\gamma$, where $S_\gamma=\{x\in\mathbb{Q}_p : |x|_p=p^\gamma\}$. Due to (2.1), $x\in S_\gamma$, $\gamma\ge 1$, if and only if $x=x_{-\gamma}p^{-\gamma}+x_{-\gamma+1}p^{-\gamma+1}+\cdots+x_{-1}p^{-1}+\xi$, where $x_{-\gamma}\ne 0$, $\xi\in B_0(0)$. Since $x_{-\gamma}p^{-\gamma}+x_{-\gamma+1}p^{-\gamma+1}+\cdots+x_{-1}p^{-1}\in I_p$, we have a "natural" decomposition of $\mathbb{Q}_p$ into a union of mutually disjoint discs:

$$\mathbb{Q}_p=\bigcup_{a\in I_p}B_0(a).$$

Therefor,

$$I_p=\{a=p^{-\gamma}\big(a_0+a_1p+\cdots+a_{\gamma-1}p^{\gamma-1}\big):$$

$$\gamma\in\mathbb{N};\ a_j=0,1,\dots,p-1;\ j=0,1,\dots,\gamma-1\} \tag{3.1}$$

is a *"natural" set of shifts* for $\mathbb{Q}_p$, which will be used in the sequel. This set I_p can be identified with the factor group $\mathbb{Q}_p/\mathbb{Z}_p$.

Let

$$J_{p;m}=\{s=p^{-m}\big(s_0+s_1p+\cdots+s_{m-1}p^{m-1}\big):$$

$$s_j=0,1,\dots,p-1;\ j=0,1,\dots,m-1; s_0\ne 0\}, \tag{3.2}$$

where $m\ge 1$ is a *fixed* positive integer.

Let us introduce the set of $(p-1)p^{m-1}$ functions

$$\theta_s^{(m)}(x)=\chi_p(sx)\Omega\big(|x|_p\big),\quad s\in J_{p;m},\quad x\in\mathbb{Q}_p, \tag{3.3}$$

and the family of functions generated by their dilatations and shifts:

$$\theta_{s;\,ja}^{(m)}(x)=p^{-j/2}\chi_p\big(s(p^jx-a)\big)\Omega\big(|p^jx-a|_p\big),\quad x\in\mathbb{Q}_p, \tag{3.4}$$

where $s\in J_{p;m}$, $j\in\mathbb{Z}$, $a\in I_p$, $\Omega(t)$ is the characteristic function of the segment $[0,1]\subset\mathbb{R}$.

Theorem 3.1. *The functions* (3.4) *form an orthonormal* non-Haar p-adic wavelet basis *in* $\mathcal{L}^2(\mathbb{Q}_p)$.

Proof. 1. Consider the scalar product

$$\big(\theta_{s';\,j'a'}^{(m)}(x),\theta_{s;\,ja}^{(m)}(x)\big)=p^{-(j+j')/2}\int_{\mathbb{Q}_p}\chi_p\big(s'(p^{j'}x-a')-s(p^{\gamma}x-a)\big)$$

$$\times\,\Omega\big(|p^jx-a|_p\big)\Omega\big(|p^{j'}x-a'|_p\big)\,dx. \tag{3.5}$$

If $j\le j'$, according to formula [35, VII.1], [24]

$$\Omega\big(|p^jx-a|_p\big)\Omega\big(|p^{j'}x-a'|_p\big)=\Omega\big(|p^jx-a|_p\big)\Omega\big(|p^{j'-j}a-a'|_p\big), \tag{3.6}$$

(3.5) can be rewritten as

$$\big(\theta^{(m)}_{s';\,j'a'}(x),\theta^{(m)}_{s;\,ja}(x)\big)=p^{-(j+j')/2}\Omega\big(|p^{j'-j}a-a'|_p\big)$$

$$\times\int_{\mathbb{Q}_p}\chi_p\big(s'(p^{j'}x-a')-s(p^{j}x-a)\big)\Omega\big(|p^{j}x-a|_p\big)\,dx. \tag{3.7}$$

Let $j<j'$. Making the change of variables $\xi=p^jx-a$ and taking into account (2.9), we obtain from (3.7)

$$\big(\theta^{(m)}_{s';\,j'a'}(x),\theta^{(m)}_{s;\,ja}(x)\big)=p^{-(j+j')/2}\chi_p\big(s'(p^{j'-j}a-a')\big)$$

$$\times\Omega\big(|p^{j'-j}a-a'|_p\big)\int_{\mathbb{Q}_p}\chi_p\big((p^{j'-j}s'-s)\xi\big)\Omega\big(|\xi|_p\big)\,d\xi$$

$$=p^{-(j+j')/2}\chi_p\big(s'(p^{j'-j}a-a')\big)$$

$$\times\,\Omega\big(|p^{j'-j}a-a'|_p\big)\Omega\big(|p^{j'-j}s'-s|_p\big). \tag{3.8}$$

Since

$$\begin{aligned} p^{j'-j}s' &= p^{j'-j-m}\big(s'_0+s'_1p+\cdots+s'_{j-1}p^{m-1}\big),\\ s &= p^{-m}\big(s_0+s_1p+\cdots+s_{j-1}p^{m-1}\big), \end{aligned}$$

where $s'_0,s_0\neq 0$, $j'-j\le 1$, it is clear that the fractional part $\{p^{j'-j}s'-s\}_p\neq 0$. Thus $\Omega\big(|p^{j'-j}s'-s|_p\big)=0$ and $\big(\theta^{(m)}_{s';\,j'a'}(x),\theta^{(m)}_{s;\,ja}(x)\big)=0$.

Consequently, the scalar product $\big(\theta^{(m)}_{s';\,j'a'}(x),\theta^{(m)}_{s;\,ja}(x)\big)=0$ can be nonzero only if $j=j'$. In this case (3.8) implies

$$\big(\theta^{(m)}_{s';\,ja'}(x),\theta^{(m)}_{s;\,ja}(x)\big)$$

$$=p^{-j}\chi_p\big(s'(a-a')\big)\Omega\big(|a-a'|_p\big)\Omega\big(|s'-s|_p\big), \tag{3.9}$$

where $\Omega\big(|a-a'|_p\big)=\delta_{a'a}$, $\Omega\big(|s'-s|_p\big)=\delta_{s's}$, and $\delta_{s's}$, $\delta_{a'a}$ are the Kronecker symbols.

Since $\int_{\mathbb{Q}_p}\Omega\big(|p^jx-a|_p\big)\,dx=p^j$, formulas (3.8), (3.9) imply that

$$\big(\theta^{(m)}_{s';\,j'a'}(x),\theta^{(m)}_{s;\,ja}(x)\big)=\delta_{s's}\delta_{j'j}\delta_{a'a}. \tag{3.10}$$

Thus the system of functions (3.4) is orthonormal.

To prove the completeness of the system of functions (3.4), we repeat the corresponding proof [24] almost word for word. Recall that the system of the characteristic functions of the discs $B_k(0)$ is complete in $\mathcal{L}^2(\mathbb{Q}_p)$. Consequently, taking into account that the system of functions $\{\theta^{(m)}_{s;\,ja}(x):s\in J_{p;m};j\in\mathbb{Z},a\in I_p\}$ is invariant under dilatations and shifts, in order to prove that it is a complete system, it is sufficient to verify the Parseval identity for the characteristic function $\Omega(|x|_p)$.

If $0\le j$, according to (3.6), (2.9),

$$\big(\Omega(|x|_p),\theta^{(m)}_{s;\,ja}(x)\big)=p^{-j/2}\Omega\big(|-a|_p\big)\int_{\mathbb{Q}_p}\chi_p\big(s(p^jx-a)\big)\Omega\big(|x|_p\big)\,dx$$

$$= p^{-j/2}\chi_p\big(-sa)\big)\Omega\big(|sp^j|_p\big)\Omega\big(|-a|_p\big)$$

$$= \begin{cases} 0, & a \neq 0, \\ 0, & a = 0, \quad j \leq m-1, \\ p^{-j/2}, & a = 0, \quad j \geq m. \end{cases} \tag{3.11}$$

If $0 > j$, according to (3.6), (2.9),

$$\big(\Omega(|x|_p), \theta^{(m)}_{s;\,ja}(x)\big) = p^{-j/2}\Omega\big(|p^{-j}a|_p\big)\int_{\mathbb{Q}_p}\chi_p\big(s(p^jx-a)\big)\Omega\big(|p^jx-a|_p\big)\,dx$$

$$= p^{j/2}\Omega\big(|p^{-j}a|_p\big)\int_{\mathbb{Q}_p}\chi_p\big(s\xi\big)\Omega\big(|\xi|_p\big)\,d\xi = p^{j/2}\Omega\big(|p^{-j}a|_p\big)\Omega\big(|s|_p\big) = 0. \tag{3.12}$$

Thus,

$$\sum_{s\in J_{p;m};j\in\mathbb{Z},a\in I_p}\Big|\big(\Omega(|x|_p), \theta^{(m)}_{s;\,ja}(x)\big)\Big|^2 = \sum_{j=m}^{\infty}\sum_{s\in J_{p;m}}p^{-j}$$

$$= p^{m-1}(p-1)\frac{p^{-m}}{1-p^{-1}} = 1 = \Big|\big(\Omega(|x|_p), \Omega(|x|_p)\big)\Big|^2.$$

Thus the system of functions (3.4) is an orthonormal basis in $\mathcal{L}^2(\mathbb{Q}_p)$.

2. Since elements of basis (3.4) can be obtained by dilatations and shifts of the set of $(p-1)p^{m-1}$ functions (3.3), it is the p-adic wavelet basis.

3. According to [32], the Haar wavelet functions are constructed by the Haar type *refinement equation* (1.4). In particular, it is easy to see that Kozyrev's wavelet functions (1.3) can be expressed in terms of the *refinable function* (1.5) as

$$\theta_k(x) = \chi_p(p^{-1}kx)\Omega\big(|x|_p\big) = \sum_{r=0}^{p-1}h_{kr}\phi\Big(\frac{1}{p}x - \frac{r}{p}\Big), \quad x \in \mathbb{Q}_p, \tag{3.13}$$

where $h_{kr} = e^{2\pi i\{\frac{kr}{p}\}_p}$, $r = 0, 1, \dots, p-1$, $k = 1, 2, \dots, p-1$. In this case the wavelet functions $\theta_k(x) = \chi_p(p^{-1}kx)\Omega\big(|x|_p\big)$ takes values in the set $\{e^{2\pi i\frac{kr}{p}} : r = 0, 1, \dots, p-1\}$ of p elements on the discs $B_{-1}(r)$, $r = 0, 1, \dots, p-1$. Thus any wavelet function $\theta_k(x)$ is represented as a linear combination of the characteristic functions of the disks of the radius of p^{-1}.

In contrast to the Kozyrev wavelet basis (1.2), the number of generating wavelet functions (3.3) for the wavelet basis (3.4) is not minimal. For example, if $p = 2$, then we have 2^{m-1} wavelet functions (instead of one as it is for (1.2) and for classical wavelet bases in real analysis.

Let $B_0 = \cup_b B_{-m}(a) \cup B_{-m}$ be the *canonical covering* (2.5) of the disc B_0 with p^m discs, $m \geq 1$, where $b = 0$ and $b = b_rp^r + b_{r+1}p^{r+1} + \cdots + b_{m-1}p^{m-1}$ is the center of the discs B_{-m} and $B_{-m}(b)$, respectively, $0 \leq b_j \leq p-1$, $j = r, r+1, \dots, m-1$, $b_r \neq 0$, $r = 0, 1, 2, \dots, m-1$.

For $x \in B_{-m}(b)$, $s \in J_{p;m}$, we have $x = b + p^m\big(y_0 + y_1p + y_2p^2 + \cdots\big)$, $s = p^{-m}\big(s_0 + s_1p + \cdots + s_{m-1}p^{m-1}\big)$, $s_0 \neq 0$; $sx = sb + \xi$, $\xi \in \mathbb{Z}_p$; and $\{sx\}_p =$

$\{sb\}_p = \{p^{r-m}\big(b_r + a_{r+1}p + \cdots + b_{m-1}p^{m-r-1}\big)\big(s_0 + s_1p + \cdots + s_{m-1}p^{m-1}\big)\}_p$, $r = 0, 1, 2, \ldots, m-1$, (see (2.2)). Thus,

$$\theta_s^{(m)}(x) = \chi_p(sx)\Omega\big(|x|_p\big)$$

$$= \begin{cases} 0, & |x|_p \ge p, \\ e^{2\pi i\{sa\}_p}, & x \in B_{-m}(b), \quad b = \sum_{l=r}^{m-1} b_l p^l, \\ 1, & x \in B_{-m}, \end{cases} \tag{3.14}$$

where $0 \le b_j \le p-1$, $j = r, \ldots, m-1$, $b_r \ne 0$, $r = 0, 1, \ldots, m-1$; $s = p^{-m}\big(s_0 + s_1p + \cdots + s_{m-1}p^{m-1}\big)$, $0 \le s_j \le p-1$, $j = 0, 1, \ldots, m-1$, $s_0 \ne 0$. Thus the function $\theta_s^{(m)}(x) = \chi_p(sx)\Omega\big(|x|_p\big)$ takes values in the set $\{e^{2\pi i\{sb\}_p} : b = \sum_{l=r}^{m-1} b_l p^l, 0 \le b_j \le p-1, j = r, \ldots, m-1, b_r \ne 0, r = 0, 1, \ldots, m-1\}$ of p^m elements on the discs $B_{-m}(b)$. By using (3.14), one can see that in contrast to the Kozyrev wavelet functions (1.3), any wavelet function $\theta_s^{(m)}(x)$ is represented as a linear combination of the characteristic functions of the disks of the radius of p^{-m}:

$$\theta_s^{(m)}(x) = \chi_p(sx)\Omega\big(|x|_p\big) = \sum_b \tilde{h}_{sb}\phi\Big(\frac{1}{p^m}x - \frac{b}{p^m}\Big), \tag{3.15}$$

$x \in \mathbb{Q}_p$, where $\tilde{h}_{sb} = e^{2\pi i\{sb\}_p}$, $b = 0$ or $b = b_r p^r + b_{r+1}p^{r+1} + \cdots + b_{m-1}p^{m-1}$, $r = 0, 1, \ldots, m-1$, $0 \le b_j \le p-1$, $b_r \ne 0$; $s \in J_{p;m}$.

According to formulas (3.14), (3.15), the wavelet function (3.3) *cannot be represented* in terms of the *Haar refinable function* $\phi(x) = \Omega\big(|x|_p\big)$ (which is a solution of the Haar *refinement equation* (1.4)), i.e., in the form

$$\theta_s^{(m)}(x) = \sum_{a \in I_p} \beta_a \phi(p^{-1}x - a), \quad \beta_a \in \mathbb{C}.$$

Consequently, the wavelet basis (3.4) is non-Haar type for $m \ge 2$. $\Box$

Making the change of variables $\xi = p^j x - a$ and taking into account (2.9), we obtain $\int \mathbb{Q}_p \theta_{s;\,ja}^{(m)}(x)\,dx = p^{j/2}\int_{\mathbb{Q}_p} \chi_p\big(s\xi\big)\Omega\big(|\xi|_p\big)\,d\xi = p^{j/2}\Omega\big(|s|_p\big) = 0$, i.e., according to Lemma 2.3, the wavelet function $\theta_{s;\,ja}^{(m)}(x)$ belongs to the Lizorkin space $\Phi(\mathbb{Q}_p)$.

Remark 3.1. 1. Kozyrev's wavelet basis (1.2) is a particular case of the wavelet basis (3.4) for $m = 1$. Indeed, in this case we have $\theta_{s;\,ja}^{(1)}(x) \equiv \theta_{k;\,ja}(x)$, where $s \equiv p^{-1}k$, $k = 1, 2, \ldots, p-1$; $j \in \mathbb{Z}$, $a \in I_p$.

2. Our non-Haar wavelet basis (3.4) can be constructed in the framework of the approach developed by J. J. Benedetto and R. L. Benedetto [8] [1]. In the notation [8], $H^\perp$ is the ball B_0, and $A_1^* : \mathbb{Q}_p \to \mathbb{Q}_p$ is multiplication by p^{-1}, and $W = (A_1^*)^{m-1}H^\perp = B_{m-1}$. In addition, the "choice coset of representatives" $\mathcal{D}$ in [8] is the set (3.1) of shifts I_p, and the set $J_{p;m}$ given by (3.2) is precisely the

[1]Here we follow the referee report of the previous version of our paper.

set $\mathcal{D}\cap\big((A_1^*W)\setminus W\big)$ that appears in equation [8, (4.1)]. The set $J_{p;m}$ consists of $N=p^m-p^{m-1}$ elements, where N is the number of wavelet generators.

The algorithm of [8] starts with N sets $\Omega s,0$ and N local translation functions T_s, one for each $s\in J_{p;m}$. In order to construct the wavelets (3.4) by using the algorithm of [8], we set

$$\Omega_{s,0}=B_0\big(p^{-m}(s_1p+\cdots+s_{m-1}p^{m-1})\big)$$

$$=p^{-m}(s_1p+\cdots+s_{m-1}p^{m-1})+B_0\subset B_{m-1},$$

i.e., remove the term s_0 and add the set $B_0=H^{\perp}$. We also define

$$T_s: B_{m-1}\to B_m\setminus B_{m-1} \quad \text{by} \quad T_s(w)=w+s_0p^{-m},$$

so that T_s maps $\Omega_{s,0}$ to $B_0(s)$ by translation. It is easy to verify that this data fit the requirements of [8]: each $\Omega_{s,0}$ is $(\tau,\mathcal{D})$-congruent to $H^{\perp}=B_0$, the union of all such sets contains a neighborhood of 0, each T_s has the form required by formula [8, (4.1)], and for each $s\neq s'$ in $J_{p;m}$, one of the two compatibility condition [8, (4.2) or (4.3)] holds. Thus, by using the algorithm of [8], one can produce the wavelets (3.4). Moreover, according to (3.28), any wavelet $\theta_s^{(m)}$ is the Fourier transform of the characteristic function of each disc $B_0(s)$, $s\in J_{p;m}$. Recall that the algorithm of [8] only allows the construction of wavelet functions whose Fourier transforms are the characteristic functions of some sets (see [8, Proposition 5.1.]).

3.2. **Countable family of non-Haar wavelet bases in $\mathcal{L}^2(\mathbb{Q}_p)$.** Now, using the proof scheme of [32, Theorem 1], we construct infinitely many different non-Haar wavelet bases, which are distinct from the basis (3.4).

In what follows, we shall write the p-adic number $a=p^{-\gamma}\big(a_0+a_1p+\cdots+a_{\gamma-1}p^{\gamma-1}\big)\in I_p$, $a_j=0,1,\dots,p-1$, $j=0,1,\dots,\gamma-1$, in the form $a=\frac{r}{p^{\gamma}}$, where $r=a_0+a_1p+\cdots+a_{\gamma-1}p^{\gamma-1}$.

Since the p-adic norm is non-Archimedean, it is easy to see that the wavelet functions (3.3) have the following property:

$$\theta_s^{(m)}(x\pm1)=\chi_p(\pm s)\theta_s^{(m)}(x), \quad s\in J_{p;m}. \tag{3.16}$$

Theorem 3.2. *Let $\nu=1,2,\dots$. The functions*

$$\psi_s^{(m)[\nu]}(x)=\sum_{k=0}^{p^{\nu}-1}\alpha_{s;k}\theta_s^{(m)}\Big(x-\frac{k}{p^{\nu}}\Big), \qquad s\in J_{p;m}, \tag{3.17}$$

are wavelet functions if and only if

$$\alpha_{s;k}=p^{-\nu}\sum_{r=0}^{p^{\nu}-1}\gamma_{s;r}e^{-2i\pi\frac{-s+r}{p}k}, \tag{3.18}$$

where $\gamma_{s;k}\in\mathbb{C}$, $|\gamma_{s;k}|=1$, $k=0,1,\dots,p^{\nu}-1$, $s\in J_{p;m}$.

Proof. Let $\psi_s^{(m)[\nu]}(x)$ be defined by (3.17), $s \in J_{p;m}$. According to Theorem 3.1, $\{\theta_s^{(m)}(\cdot - a), s \in J_{p;m}, a \in I_p\}$ is an orthonormal system. Hence, taking into account (3.16), we see that $\psi_s^{(m)[\nu]}$ is orthogonal to $\psi_s^{[\nu]}(x)(\cdot - a)$ whenever $a \in I_p$, $a \neq \frac{k}{p^\nu}$, $k = 0, 1, \dots p^\nu - 1$; $\nu = 1, 2, \dots$. Thus the system $\{\psi_s^{(m)[\nu]}(x - a), s \in J_{p;m}, a \in I_p\}$ is orthonormal if and only if the system consisting of the functions

$$\psi_s^{(m)[\nu]}\Big(x - \frac{r}{p^\nu}\Big) = \chi_p(-s)\alpha_{s;p^\nu-r}\theta_s^{(m)}(x) + \chi_p(-s)\alpha_{s;p^\nu-r+1}\theta_s^{(m)}\Big(x - \frac{1}{p^\nu}\Big)$$
$$+ \dots + \chi_p(-s)\alpha_{s;p^\nu-1}\theta_s^{(m)}\Big(x - \frac{r-1}{p^\nu}\Big)$$
$$+\alpha_{s;0}\theta_s^{(m)}\Big(x - \frac{r}{p^\nu}\Big) + \alpha_{s;1}\theta_s^{(m)}\Big(x - \frac{r+1}{p^\nu}\Big)$$
$$+ \dots + \alpha_{s;p^\nu-1-r}\theta_s^{(m)}\Big(x - \frac{p^\nu-1}{p^\nu}\Big), \tag{3.19}$$

$r = 0, \dots, p^\nu - 1$, $s \in J_{p;m}$, is orthonormal, $\nu = 1, 2, \dots$. Set

$$\Xi_\nu^{[0]} = \left(\theta_s^{(m)}, \theta_s^{(m)}\left(\cdot - \frac{1}{p^\nu}\right), \dots, \theta_s^{(m)}\left(\cdot - \frac{p^\nu - 1}{p^\nu}\right)\right)^T,$$
$$\Xi_\nu^{[\nu]} = \left(\psi_s^{(m)[\nu]}, \psi_s^{(m)[\nu]}\left(\cdot - \frac{1}{p^\nu}\right), \dots, \psi_s^{(m)[\nu]}\left(\cdot - \frac{p^\nu - 1}{p^\nu}\right)\right)^T,$$

where T is the transposition operation. By (3.19), we have $\Xi_s^{[\nu]} = D_s\Xi_s^{[0]}$, where

$$D_s =$$
$$\begin{pmatrix} \alpha_{s;0} & \alpha_{s;1} & \dots & \alpha_{s;p^\nu-2} & \alpha_{s;p^\nu-1} \\ \chi_p(-s)\alpha_{s;p^\nu-1} & \alpha_{s;0} & \dots & \alpha_{s;p^\nu-3} & \alpha_{s;p^\nu-2} \\ \chi_p(-s)\alpha_{s;p^\nu-2} & \chi_p(-s)\alpha_{s;p^\nu-1} & \dots & \alpha_{s;p^\nu-4} & \alpha_{s;p^\nu-3} \\ \dots & \dots & \dots & \dots & \dots \\ \chi_p(-s)\alpha_{s;2} & \chi_p(-s)\alpha_{s;3} & \dots & \alpha_{s;0} & \alpha_{s;1} \\ \chi_p(-s)\alpha_{s;1} & \chi_p(-s)\alpha_{s;2} & \dots & \chi_p(-s)\alpha_{s;p^\nu-1} & \alpha_{s;0} \end{pmatrix}, \tag{3.20}$$

and $s \in J_{p;m}$; $\nu = 1, 2, \dots$. Due to orthonormality of $\{\psi_s^{(m)[\nu]}(x)(\cdot - a), s \in J_{p;m}, a \in I_p\}$, the coordinates of $\Xi_s^{[\nu]}$ form an orthonormal system if and only if the matrixes D_s are unitary, $s \in J_{p;m}$.

Let $u_s = (\alpha_{s;0}, \alpha_{s;1}, \dots, \alpha_{s;p^\nu-1})^T$ be a vector and let

$$A_s = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \chi_p(-s) \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

be a $p^\nu \times p^\nu$ matrix, $s \in J_{p;m}$. It is not difficult to see that

$$A_s^r u_\nu = \big(\chi_p(-s)\alpha_{s;p^\nu-r}, \chi_p(-s)\alpha_{s;p^\nu-r+1}, \dots, \chi_p(-s)\alpha_{s;p^\nu-1},$$

$$\alpha_{s;0}, \alpha_{s;1}, \dots, \alpha_{s;p^\nu-r-1}\big)^T,$$

where $r = 0, 1, \dots, p^s - 1,\ s \in J_{p;m};\ \nu = 1, 2, \dots$. Thus we have

$$D_s = \big(u_s, A_s u_s, \dots, A_s^{p^\nu-1} u_s\big)^T.$$

Consequently, to describe all unitary matrixes D_s, we should find all vectors $u_s = (\alpha_{s;0}, \alpha_{s;1}, \dots, \alpha_{s;p^\nu-1})^T$ such that the system of vectors $\{A_s^r u_s, r = 0, \dots, p^\nu - 1\}$ is orthonormal, $s \in J_{p;m}$. We already have such a vector $u_0 = (1, 0, \dots, 0, 0)^T$ because the matrix $D_0 = \big(u_0, Au_0, \dots, A^{p^\nu-1}u_0\big)^T$ is the identity matrix.

Now we prove that the system $\{A_s^r u_s, r = 0, \dots, p^\nu - 1\}$ is orthonormal if and only if $u_s = B_s u_0$, where B_s is a unitary matrix such that $A_s B_s = B_s A_s,\ s \in J_{p;m}$. Indeed, if $u_s = B_s u_0$, B_s is a unitary matrix such that $A_s B_s = B_s A_s$, then $A_s^r u_s = B_s A_s^r u_0,\ \ r = 0, 1, \dots, p^\nu - 1,\ s \in J_{p;m}$. Since the system $\{A_s^r u_0, r = 0, 1, \dots, p^\nu - 1\}$ is orthonormal and the matrix B_s is unitary, the vectors $A_s^r u_s,\ r = 0, 1, \dots, p^\nu - 1$ are also orthonormal, $s \in J_{p;m}$. Conversely, if the system $A_s^r u_s,\ \ r = 0, 1, \dots, p^\nu - 1$ is orthonormal, taking into account that $\{A_s^r u_0, r = 0, 1, \dots, p^\nu - 1\}$ is also an orthonormal system, we conclude that there exists a unitary matrix B_s such that $A_s^r u_s = B_s(A_s^r u_0)$, $r = 0, 1, \dots, p^\nu - 1$. Since $A_s^{p^\nu} u_s = \chi_p(-s)u_s$ and $A_s^{p^\nu} u_0 = \chi_p(-s)u_0$, we obtain additionally $A_s^{p^\nu} u_s = B_s A_s^{p^\nu} u_0$. It follows from the above relations that $(A_s B_s - B_s A_s)(A_s^r u_0) = 0,\ \ r = 0, 1, \dots, p^\nu - 1$. Since the vectors $A_s^r u_0$, $r = 0, 1, \dots, p^\nu - 1$ form a basis in the p^ν-dimensional space, we conclude that $A_s B_s = B_s A_s,\ s \in J_{p;m}$.

Thus all required unitary matrixes (3.20) are given by

$$D_s = \big(B_s u_0, B_s A_\nu u_0, \dots, B_s A_s^{p^\nu-1} u_0\big)^T,$$

where $u_0 = (1, 0, \dots, 0, 0)^T$ and B_s is a unitary matrix such that $A_s B_s = B_s A_s$, $s \in J_{p;m}$. It remains to describe all such matrixes B_s.

It is easy to see that the eigenvalues of A_s and the corresponding normalized eigenvectors are respectively

$$\lambda_{s;r} = e^{2i\pi\frac{-s+r}{p^\nu}} \tag{3.21}$$

and

$$v_{s;r} = \big((v_{s;r})_0, \dots, (v_{s;r})_{p^\nu-1}\big)^T,$$

where

$$(v_{s;r})_l = p^{-\nu/2} e^{-2i\pi\frac{-s+r}{p^\nu}l}, \quad l = 0, 1, 2, \dots, p^\nu - 1, \tag{3.22}$$

$r=0,1,\dots,p^{\nu}-1$, $s\in J_{p;m}$. Hence the matrix A_s can be represented as $A_s=C_s\widetilde{A}_sC_s^{-1}$, where

$$\widetilde{A}_s=\begin{pmatrix}\lambda_0 & 0 & \dots & 0\\ 0 & \lambda_1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_{p^{\nu}-1}\end{pmatrix}$$

is a diagonal matrix, $C_s=\big(v_{s;0},\dots,v_{s;p^{\nu}-1}\big)$ is a unitary matrix. It follows that the matrix $B_s=C_s\widetilde{B}_sC_s^{-1}$ is unitary if and only if $\widetilde{B}_s$ is unitary. On the other hand, $A_sB_s=B_sA_s$ if and only if $\widetilde{A}_s\widetilde{B}_s=\widetilde{B}_s\widetilde{A}_s$. Moreover, since according to (3.21), $\lambda_{s;k}\ne\lambda_{s;l}$ whenever $k\ne l$, all unitary matrixes $\widetilde{B}_s$ such that $\widetilde{A}_s\widetilde{B}_s=\widetilde{B}_s\widetilde{A}_s$, are given by

$$\widetilde{B}_s=\begin{pmatrix}\gamma_{s;0} & 0 & \dots & 0\\ 0 & \gamma_{s;1} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \gamma_{s;p^{\nu}-1}\end{pmatrix},$$

where $\gamma_{s;k}\in\mathbb{C}$, $|\gamma_{s;k}|=1$, $k=0,1,\dots,p^{\nu}-1$, $s\in J_{p;m}$. Hence all unitary matrixes B_s such that $A_sB_s=B_sA_s$, are given by $B_s=C\widetilde{B}_sC_s^{-1}$. Using (3.22), one can calculate

$$\begin{aligned}\alpha_{s;k}&=(B_su_0)_k=(C_s\widetilde{B}_sC_s^{-1}u_0)_k\\ &=\sum_{r=0}^{p^{\nu}-1}\gamma_{s;r}(v_{s;r})_k(\overline{v}_{s;r})_0=p^{-\nu}\sum_{r=0}^{p^{\nu}-1}\gamma_{\nu;r}e^{-2i\pi\frac{-s+r}{p^{\nu}}k},\end{aligned}$$

where $\gamma_{s;k}\in\mathbb{C}$, $|\gamma_{s;k}|=1$, $k=0,1,\dots,p^{\nu}-1,$, $s\in J_{p;m}$.

It remains to prove that

$$\{p^{-j/2}\psi_s^{(m)[\nu]}(p^jx-a),x\in\mathbb{Q}_p:s\in J_{p;m},\,j\in\mathbb{Z},\,a\in I_p\}\tag{3.23}$$

is a basis for $\mathcal{L}^2(\mathbb{Q}_p)$ whenever $\psi_s^{(m)[\nu]}$ is defined by (3.17), (3.18), $\nu=1,2,\dots$. Since according to Theorem 3.1,

$$\{p^{-j/2}\theta_s^{(m)}(p^jx-a),x\in\mathbb{Q}_p:s\in J_{p;m},\,j\in\mathbb{Z},\,a\in I_p\}$$

is a basis for $\mathcal{L}^2(\mathbb{Q}_p)$, it suffices to check that any function $p^{-j/2}\theta_s^{(m)}(p^jx-c)$, $c\in I_p$, can be decomposed with respect to the functions $p^{-j/2}\psi_s^{(m)[\nu]}(p^jx-a)$, $a\in I_p$; where $s\in J_{p;m}$, $j\in\mathbb{Z}$. Any $c\in I_p$, $c\ne0$, can be represented in the form $c=\frac{r}{p^{\nu}}+b$, where $r=0,1,\dots,p^{\nu}-1$, $|b|_p\ge p^{\nu+1}$, $bp^{\nu}\in I_p$. Taking into account that $\Xi_s^{[0]}=D_s^{-1}\Xi_s^{[\nu]}$, i.e.,

$$\theta_s^{(m)}\Big(p^jx-\frac{r}{p^{\nu}}\Big)=\sum_{k=0}^{p^{\nu}-1}\beta_{s;k}^{(r)}\psi_s^{(m)[\nu]}\Big(p^jx-\frac{k}{p^{\nu}}\Big),\quad r=0,1,\dots,p^{\nu}-1,$$

we have

$$\theta_s^{(m)}\Big(p^j x - c\Big) = \theta_s^{(m)}\Big(p^j x - \frac{r}{p^\nu} - b\Big) = \sum_{k=0}^{p^\nu-1} \beta_{s;k}^{(r)} \psi_s^{(m)[\nu]}\Big(p^j x - \frac{k}{p^\nu} - b\Big),$$

and $\frac{k}{p^\nu} + b \in I_p$, $k = 0, 1, \dots, p^\nu - 1$, $\nu = 1, 2, \dots$. Consequently, any function $f \in \mathcal{L}^2(\mathbb{Q}_p)$ can be decomposed with respect to the system of functions (3.23). □

Thus, we have constructed a countable family of non-Haar wavelet bases given by formulas (3.23), (3.17), (3.18).

Remark 3.2. Let $\nu = 1$. According to (3.23), (3.17), (3.18); (3.3), (3.4), in this case we have

$$\psi_s^{(m)[1]}(x-a) = \sum_{k=0}^{p-1} \alpha_{s;k} \theta_s^{(m)}\Big(x - a - \frac{k}{p}\Big), \quad s \in J_{p;m}, \quad a \in I_p.$$

Applying formulas (3.28), (2.8) to the last relation, we obtain

$$F[\psi_s^{(m)[1]}(x-a)](\xi) = \chi_p(a\xi)\Omega\big(|\xi + s|_p\big) \sum_{k=0}^{p-1} \alpha_{s;k} \chi_p\Big(\frac{k}{p}\xi\Big), \quad s \in J_{p;m}, \quad a \in I_p.$$

Thus the right-hand side of the last relation is not equal to zero only if $\xi = -s + \eta$, $\eta = \eta_0 + \eta_1 p + \cdots \in \mathbb{Z}_p$, i.e., $\xi \in B_0(-s)$. Hence, for $\xi \in B_0(-s)$ we have

$$F[\psi_s^{(m)[1]}(x-a)](\xi) = \chi_p(-as)\chi_p(a\eta) \sum_{k=0}^{p-1} \alpha_{s;k} \chi_p\Big(\frac{k}{p}(-s + \eta_0)\Big), \quad \eta \in \mathbb{Z}_p,$$

where $s \in J_{p;m}$, $a \in I_p$,. It is easy to verify that for any $s \in J_{p;m}$ and $a \in I_p$ the right-hand side of last relation *is not a characteristic function of any set*. According to [8, Proposition 5.1.], *only* the functions whose Fourier transforms are the characteristic functions of some sets may be wavelet functions obtained by Benedettos' method. Consequently, the wavelet basis corresponding to the generating wavelet functions $\psi_s^{(m)[1]}$, $s \in J_{p;m}$, *cannot be constructed* by the method of [8].

3.3. **Multidimensional non-Haar p-adic wavelets.** Since the one- dimensional wavelets (3.4) are non-Haar type, we cannot construct the n-dimensional wavelet basis as the tensor products of the one-dimensional MRAs (see [32]). In this case we introduce n-dimensional non-Haar wavelet functions as the n-direct product of the one-dimensional non-Haar wavelets (3.4). For $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$ and $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$ we introduce a multi-dilatation

$$\widehat{p^j} x \stackrel{def}{=} (p^{j_1} x_1, \dots, p^{j_n} x_n) \tag{3.24}$$

and define the n-direct products of the one-dimensional p-adic wavelets (3.4) as

$$\Theta^{(m)\times}_{s;\,ja}(x) = p^{-|j|/2}\chi_p\big(s\cdot(\widehat{p^{j}}x-a)\big)\Omega\big(|\widehat{p^{j}}x-a|_p\big), \quad x\in\mathbb{Q}_p^n, \tag{3.25}$$

where $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$; $|j|=j_1+\cdots+j_n$; $a=(a_1,\dots,a_n)\in I_p^n$; $s=(s_1,\dots,s_n)\in J_{p;m}^n$; $m=(m_1,\dots,m_n)$, $m_l\ge 1$ is a *fixed* positive integer, $l=1,2,\dots,n$. Here I_p^n, $J_{p;m}^n$ are the n- direct products of the corresponding sets (3.1) and (3.2).

In view of (2.4), Theorem 3.1 implies the following statement.

Theorem 3.3. *The non-Haar wavelet functions* (3.25) *form an orthonormal basis in* $\mathcal{L}^2(\mathbb{Q}_p^n)$.

Using (2.4), it is easy to verify that $\int_{\mathbb{Q}_p^n}\Theta^{(m)\times}_{s;\,ja}(x)\,d^nx=0$, i.e., in view of Lemma 2.3, the wavelet function $\Theta^{(m)\times}_{s;\,ja}(x)$, $j\in\mathbb{Z}^n$, $a\in I_p^n$, $s\in J_{p;m}^n$, belongs to the Lizorkin space $\Phi(\mathbb{Q}_p^n)$.

Corollary 3.1. *The n-direct product of one-dimensional Kozyrev's wavelets* (1.2)

$$\Theta^{\times}_{k;ja}(x) = p^{-|j|/2}\chi_p\big(p^{-1}k\cdot(\widehat{p^{j}}x-a)\big)\Omega\big(|\widehat{p^{j}}x-a|_p\big), \quad x\in\mathbb{Q}_p^n, \tag{3.26}$$

form an orthonormal complete basis in $\mathcal{L}^2(\mathbb{Q}_p^n)$, $k\in J_p^n$, $j\in\mathbb{Z}^n$, $a\in I_p^n$.

The proof follows from Theorem 3.3 if we set $m=1$.

Corollary 3.2. *The family of functions*

$$\widetilde{\Theta}^{(m)\times}_{s;\,ja}(\xi) = F[\Theta^{(m)\times}_{s;\,ja}](\xi) = p^{|j|/2}\chi_p\big(\widehat{p^{-j}}a\cdot\xi\big)\Omega\big(|s+\widehat{p^{-j}}\xi|_p\big), \quad \xi\in\mathbb{Q}_p^n, \tag{3.27}$$

form an orthonormal complete basis in $\mathcal{L}^2(\mathbb{Q}_p^n)$, $j\in\mathbb{Z}^n$; $a\in I_p^n$; $s\in J_{p;m}^n$; $m=(m_1,\dots,m_n)$, $m_l\ge 1$ *is a fixed positive integer*, $l=1,2,\dots,n$.

Proof. Consider the function $\Theta^{(m)\times}_{s}(x)=\chi_p(s\cdot x)\Omega(|x|_p)$ generated by the direct product of functions (3.3), $x\in\mathbb{Q}_p^n$, $s=(s_1,\dots,s_n)\in J_{p;m}^n$, $s_k\in J_{p;m_k}$, $k=1,2,\dots,n$. Using (2.4), (2.9), (2.8), we have

$$F[\Theta^{(m)\times}_{s}(x)](\xi) = F\Big[\prod_{k=1}^{n}\chi_p(x_ks_k)\Omega(|x_k|_p)\Big](\xi) = \prod_{k=1}^{n}F\Big[\Omega(|x_k|_p)\Big](\xi_k+s_k|_p)$$

$$= \prod_{k=1}^{n}\Omega\big(|\xi_k+s_k|_p\big) = \Omega\big(|\xi+s|_p\big), \quad \xi\in\mathbb{Q}_p^n. \tag{3.28}$$

Here, in view of (2.4), $\Omega\big(|\xi+s|_p\big)=\Omega\big(|\xi_1+s_1|_p\big)\times\cdots\times\Omega\big(|\xi_n+s_n|_p\big)$. According to (3.2), $|s_k|_p=p^{m_k}$ and $\Omega\big(|\xi_k+s_k|_p\big)\ne 0$ only if $\xi_k=-s_k+\eta_k$, where $\eta_k\in\mathbb{Z}_p$, $s_k\in J_{p;m_k}$, $k=1,2,\dots,n$. This yields $\xi=-s+\eta$, where $\eta\in\mathbb{Z}_p^n$, $s\in J_{p;m}^n$, and in view of (2.3), $|\xi|_p=p^{\max\{m_1,\dots,m_n\}}$.

In view of formulas (3.25), (3.28), (2.8), we have

$$F[\Theta_{s;\,ja}^{(m)\times}(x)](\xi) = p^{-|j|/2}F[\Theta_{s}^{(m)\times}(\widehat{p^{j}}x-a)](\xi)$$
$$= p^{|j|/2}\chi_p\big(\widehat{p^{-j}}a\cdot\xi\big)\Omega\big(|s+\widehat{p^{-j}}\xi|_p\big),$$

i.e., (3.27).

The formula (3.27), the Parseval formula [35, VII,(4.1)], and Theorem 3.3 imply the statement. □

Similarly, one can construct n-dimensional non-Haar wavelet bases generated by the one-dimensional non-Haar wavelets (3.23) (as the n-direct product):

$$\Psi_{s;\,ja}^{(m)[\nu]\times}(x) = p^{-|j|/2}\psi_{s}^{(m_1)[\nu]}(p^{j_1}x_1-a_1)\cdots\psi_{s}^{(m_n)[\nu]}(p^{j_n}x_n-a_n), \tag{3.29}$$

where $x\in\mathbb{Q}_p^n$ and $\psi_{s_{j_k}}^{(m_{j_k})[\nu]}$ is defined by (3.17), (3.18), $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$; $|j| = j_1+\cdots+j_n$; $a=(a_1,\dots,a_n)\in I_p^n$; $s=(s_1,\dots,s_n)\in J_{p;m}^n$; $m=(m_1,\dots,m_n)$, $m_k\ge 1$ is a *fixed* positive integer, $k=1,2,\dots,n$; $\nu=1,2,\dots$.

3.4. **p-Adic Lizorkin spaces and wavelets.** Now we prove an analog of Lemma 2.1 for the Lizorkin test functions from $\Phi(\mathbb{Q}_p^n)$.

Lemma 3.1. *Any function $\phi\in\Phi(\mathbb{Q}_p^n)$ can be represented in the form of a* finite *sum*

$$\phi(x) = \sum_{s\in J_{p;m}^n,\, j\in\mathbb{Z}^n,\, a\in I_p^n} c_{s;\,j,a}\Theta_{s;\,ja}^{(m)\times}(x), \quad x\in\mathbb{Q}_p^n, \tag{3.30}$$

where $c_{s;\,j,a}$ are constants; $\Theta_{s;\,ja}^{(m)\times}(x)$ are elements of the non-Haar wavelet basis (3.25); $s=(s_1,\dots,s_n)\in J_{p;m}^n$; $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$, $|j|=j_1+\cdots+j_n$; $a=(a_1,\dots,a_n)\in I_p^n$; $m=(m_1,\dots,m_n)$, $m_l\ge 1$ is a fixed positive integer, $l=1,2,\dots,n$.

Proof. Let us calculate $\mathcal{L}^2(\mathbb{Q}_p^n)$-scalar product $(\phi(x),\Theta_{s;\,ja}^{(m)\times}(x))$. Taking into account formula (3.27) and using the Parseval-Steklov theorem, we obtain

$$c_{s;\,j,a} = \big(\phi(x),\Theta_{s;\,ja}^{(m)\times}(x)\big) = \big(F[\phi](\xi),F[\Theta_{s;\,ja}^{(m)\times}](\xi)\big)$$
$$= \big(\psi(\xi),p^{|j|/2}\chi_p\big(\widehat{p^{-j}}a\cdot\xi\big)\Omega\big(|s+\widehat{p^{-j}}\xi|_p\big)\big), \tag{3.31}$$

where $j\in\mathbb{Z}^n$, $a\in I_p^n$, $s\in J_{p;m}^n$. Here, according to Lemma 2.3, any function $\phi\in\Phi(\mathbb{Q}_p^n)$ belongs to one of the spaces $\mathcal{D}_N^l(\mathbb{Q}_p^n)$, $\psi=F^{-1}[\phi]\in\Psi(\mathbb{Q}_p^n)\cap\mathcal{D}_{-l}^{-N}(\mathbb{Q}_p^n)$, and $\operatorname{supp}\psi\subset B_{-l}^n\setminus B_{-N}^n$.

Let $|s|_p\ne|\widehat{p^{-j}}\xi|_p$. Since $p^{-N}\le|\xi|_p\le p^{-l}$ and

$$|s+\widehat{p^{-j}}\xi|_p = \max(|s|_p,|\widehat{p^{-j}}\xi|_p) = \max\big(p^{\max(m_1,\dots,m_n)},\max(p^{j_k}|\xi_k|_p\big),$$

in view of (3.31), it is clear that there are finite quantity of indexes $s=(s_1,\dots,s_n)\in J_{p;m}^n$, $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$ such that $c_{s;\,j,a}\ne 0$. The case $|s|_p=|\widehat{p^{-j}}\xi|_p=p^{\max(m_1,\dots,m_n)}$ can be considered in the same way.

Thus equality (3.30) holds in the sense of $\mathcal{L}^2(\mathbb{Q}_p^n)$. Consequently, it holds in the usual sense. □

Using standard results from the book [31] or repeating the reasoning [20], [21] almost word for word, we obtain the following assertion.

Proposition 3.1. *Any distribution $f \in \Phi'(\mathbb{Q}_p^n)$ can be realized in the form of an* infinite *sum of the form*

$$f(x) = \sum_{s\in J_{p;m}^n, j\in\mathbb{Z}^n, a\in I_p^n} d_{s;\,j,a}\Theta_{s;\,ja}^{(m)\times}(x), \quad x \in \mathbb{Q}_p^n, \tag{3.32}$$

where $d_{s;\,j,a}$ are constants; $\Theta_{s;\,ja}^{(m)\times}(x)$ are elements of the non-Haar wavelet basis (3.25); $s = (s_1, \dots, s_n) \in J_{p;m}^n$; $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$, $|j| = j_1 + \cdots + j_n$; $a = (a_1, \dots, a_n) \in I_p^n$; $m = (m_1, \dots, m_n)$, $m_l \geq 1$ is a fixed positive integer, $l = 1, 2, \dots, n$.

Here any distribution $f \in \Phi'(\mathbb{Q}_p^n)$ is associated with the representation (3.32), where the coefficients

$$d_{s;\,j,a} \stackrel{def}{=} \big\langle f, \Theta_{s;\,ja}^{(m)\times} \big\rangle, \quad s \in J_{p;m}^n, \quad j \in \mathbb{Z}^n, \quad a \in I_p^n. \tag{3.33}$$

And vice versa, taking into account Lemma 3.1 and orthonormality of the wavelet basis (3.25), any infinite sum is associated with the distribution $f \in \Phi'(\mathbb{Q}_p^n)$ whose action on a test function $\phi \in \Phi(\mathbb{Q}_p^n)$ is defined as

$$\langle f, \phi \rangle = \sum_{s\in J_{p;m}^n, j\in\mathbb{Z}^n, a\in I_p^n} d_{s;\,j,a} c_{s;\,j,a}, \tag{3.34}$$

where the sum is finite.

It is clear that in Lemma 3.1 and Proposition 3.1 instead of the basis (3.4) or its multidimensional generalization (3.25), one can use the bases (3.23), (3.17), (3.18) and their multidimensional generalizations.

In [5], the assertions of the type of Lemma 3.1 and Proposition 3.1 were stated for ultrametric Lizorkin spaces.

4. Spectral theory of p-adic pseudo-differential operators

4.1. **Pseudo-differential operators in the Lizorkin spaces.** In this subsection we present some facts on pseudo-differential operators which were introduced in [2], [3]. Consider a class of pseudo-differential operators in the Lizorkin space $\Phi(\mathbb{Q}_p^n)$

$$\begin{aligned}(A\phi)(x) &= F^{-1}\big[\mathcal{A}(\cdot)\, F[\phi](\cdot)\big](x) \\ &= \int_{\mathbb{Q}_p^n}\int_{\mathbb{Q}_p^n} \chi_p\big((y-x)\cdot\xi\big)\mathcal{A}(\xi)\phi(y)\, d^n\xi\, d^n y, \quad \phi \in \Phi(\mathbb{Q}_p^n)\end{aligned} \tag{4.1}$$

with symbols $\mathcal{A} \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$.

Remark 4.1. The class of operators (4.1) includes the Taibleson fractional operator with the symbol of the form $|\xi|_p^\alpha$ (see (4.4)); the Kochubei operator with the symbol of the form $\mathcal{A}(\xi)=|f(\xi_1,\dots,\xi_n)|_p^\alpha$, $\alpha>0$, where $f(\xi_1,\dots,\xi_n)$ is a quadratic form such that $f(\xi_1,\dots,\xi_n)\neq 0$ when $|\xi_1|_p+\cdots|\xi_n|_p\neq 0$ (see [22], [23]); the Zuniga-Galindo operator with the symbol of the form $\mathcal{A}(\xi)=|f(\xi_1,\dots,\xi_n)|_p^\alpha$, $\alpha>0$, where $f(\xi_1,\dots,\xi_n)$ is a non-constant polynomial (see [38], [39]).

If we define a conjugate pseudo-differential operator A^T as

$$(A^T\phi)(x)=F^{-1}[\mathcal{A}(-\xi)F[\phi](\xi)](x)=\int_{\mathbb{Q}_p^n}\chi_p(-x\cdot\xi)\mathcal{A}(-\xi)F[\phi](\xi)\,d^n\xi$$

then one can define the operator A in the Lizorkin space of distributions: for $f\in\Phi'(\mathbb{Q}_p^n)$ we have

$$\langle Af,\phi\rangle=\langle f,A^T\phi\rangle,\qquad\forall\,\phi\in\Phi(\mathbb{Q}_p^n). \tag{4.2}$$

Lemma 4.1. ([2]) *The Lizorkin spaces $\Phi(\mathbb{Q}_p^n)$ and $\Phi'(\mathbb{Q}_p^n)$ are invariant under the pseudo-differential operators* (4.1)*.*

Proof. In view of (2.7) and results of Subsec. 2.2, both functions $F[\phi](\xi)$ and $\mathcal{A}(\xi)F[\phi](\xi)$ belong to $\Psi(\mathbb{Q}_p^n)$, and, consequently, $(A\phi)(x)\in\Phi(\mathbb{Q}_p^n)$, i.e., $A(\Phi(\mathbb{Q}_p^n))\subset\Phi(\mathbb{Q}_p^n)$. Thus the pseudo-differential operators (4.1) are well defined, and the Lizorkin space $\Phi(\mathbb{Q}_p^n)$ is invariant under them. In view of (4.2), the Lizorkin space of distributions $\Phi'(\mathbb{Q}_p^n)$ is also invariant under pseudo-differential operator A. □

4.2. **The Taibleson fractional operator in the Lizorkin spaces.** In particular, setting $\mathcal{A}(\xi)=|\xi|_p^\alpha$, $\xi\in\mathbb{Q}_p^n$, we obtain the multi-dimensional Taibleson fractional operator. This operator was introduced in [33, §2], [34, III.4.] on the space of distributions $\mathcal{D}'(\mathbb{Q}_p^n)$ for $\alpha\in\mathbb{C}$, $\alpha\neq -n$. Next, in [2], the Taibleson fractional operator was defined and studied in the Lizorkin space of distributions $\Phi'(\mathbb{Q}_p^n)$ for all $\alpha\in\mathbb{C}$. According to (4.1), (4.2),

$$\big(D^\alpha f\big)(x)=F^{-1}\big[|\cdot|_p^\alpha F[f](\cdot)\big](x),\quad f\in\Phi'(\mathbb{Q}_p^n). \tag{4.3}$$

Representation (4.3) can be rewritten as a convolution

$$\big(D^\alpha f\big)(x)=\kappa_{-\alpha}(x)*f(x)=\langle\kappa_{-\alpha}(x),f(x-\xi)\rangle,\quad f\in\Phi'(\mathbb{Q}_p^n),\quad\alpha\in\mathbb{C}, \tag{4.4}$$

where according to [2], the multidimensional *Riesz kernel* is given by the formula

$$\kappa_\alpha(x)=\begin{cases}\frac{|x|_p^{\alpha-n}}{\Gamma_p^{(n)}(\alpha)}, & \alpha\neq 0,\ n,\\ \delta(x) & \alpha=0,\\ -\frac{1-p^{-n}}{\log p}\log|x|_p & \alpha=n\end{cases}$$

the function $|x|_p$, $x\in\mathbb{Q}_p^n$ is defined by (2.3). Here the multidimensional homogeneous distribution $|x|_p^{\alpha-n}\in\mathcal{D}'(\mathbb{Q}_p^n)$ of degree $\alpha-n$ was defined in [33,

(*)], [34, III,(4.3)], [35, VIII,(4.2)], $\Gamma_p^{(n)}(\alpha)$ is the n-dimensional Γ-*function* defined in [33, Theorem 1.], [34, III,Theorem (4.2)], [35, VIII,(4.4)].

According to Lemma 4.1 and (4.1), (4.2), the Lizorkin space $\Phi'(\mathbb{Q}_p^n)$ is invariant under the Taibleson fractional operator D^α for all $\alpha\in\mathbb{C}$ [2].

4.3. **p-Adic wavelets as eigenfunctions of pseudo-differential operators.** As mentioned above in Sec. 1, it is typical that p-adic compactly supported wavelets are eigenfunctions of p-adic pseudo-differential operators. For example, in [24] S. V. Kozyrev proved that wavelets (1.2) are eigenfunctions of the one-dimensional fractional operator (4.3), (4.4) for $\alpha>0$:

$$D^\alpha\theta_{k;\,ja}(x)=p^{\alpha(1-j)}\theta_{k;\,ja}(x),\quad x\in\mathbb{Q}_p, \tag{4.5}$$

where $k=1,2,\dots p-1$, $j\in\mathbb{Z}$, $a\in I_p$. Since wavelet functions (1.2) belong to the Lizorkin space, the relation (4.5) holds for all $\alpha\in\mathbb{C}$.

Now we study the spectral problem for pseudo-differential operators (4.1) in connection with the wavelet functions (3.25) and (3.29).

Theorem 4.1. *Let A be a pseudo-differential operator* (4.1) *with a symbol $\mathcal{A}(\xi)\in\mathcal{E}(\mathbb{Q}_p^n\setminus\{0\})$. Then the n-dimensional non-Haar wavelet function* (3.25)

$$\Theta_{s;\,ja}^{(m)\times}(x)=p^{-|j|/2}\chi_p\big(s\cdot(\widehat{p^j}x-a)\big)\Omega\big(|\widehat{p^j}x-a|_p\big),\quad x\in\mathbb{Q}_p^n,$$

is an eigenfunction of A if and only if

$$\mathcal{A}\big(\widehat{p^j}(-s+\eta)\big)=\mathcal{A}\big(-\widehat{p^j}s\big),\qquad\forall\,\eta\in\mathbb{Z}_p^n, \tag{4.6}$$

where $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$; $a\in I_p^n$; $s\in J_{p;m}^n$; and $m=(m_1,\dots,m_n)$, $m_j\ge1$ is a fixed positive integer, $j=1,2,\dots,n$. The corresponding eigenvalue is $\lambda=\mathcal{A}\big(-\widehat{p^j}s\big)$, i.e.,

$$A\Theta_{s;\,ja}^{(m)\times}(x)=\mathcal{A}(-\widehat{p^j}s)\Theta_{s;\,ja}^{(m)\times}(x).$$

Here the multi-dilatation is defined by (3.24), *I_p^n and $J_{p;m}^n$ are the n-direct products of the corresponding sets* (3.1) *and* (3.2).

Proof. Let condition (4.6) be satisfied. Then (4.1) and the above formula (3.27) imply that

$$\begin{aligned}A\Theta_{s;\,ja}^{(m)\times}(x)&=F^{-1}\big[\mathcal{A}(\xi)F[\Theta_{s;\,ja}^{(m)\times}](\xi)\big](x)\\&=p^{|j|/2}F^{-1}\big[\mathcal{A}(\xi)\chi_p\big(\widehat{p^{-j}}a\cdot\xi\big)\Omega\big(|s+\widehat{p^{-j}}\xi|_p\big)\big](x).\end{aligned}\tag{4.7}$$

Making the change of variables $\xi=\widehat{p^j}(\eta-s)$ and using (2.9), we obtain

$$\begin{aligned}A\Theta_{s;\,ja}^{(m)\times}(x)&=p^{-|j|/2}\int_{\mathbb{Q}_p^n}\chi_p\big(-(\widehat{p^j}x-a)\cdot(\eta-s)\big)\mathcal{A}(\widehat{p^j}(\eta-s))\,\Omega(|\eta|_p)\,d^n\eta\\&=p^{-|j|/2}\mathcal{A}(-\widehat{p^j}s)\chi_p\big(s\cdot(\widehat{p^j}x-a)\big)\int_{B_0^n}\chi_p(-(\widehat{p^j}x-a)\cdot\eta)\,d^n\eta\end{aligned}$$

$$= \mathcal{A}(-\widehat{p^{j}}s)\Theta_{s;\,ja}^{(m)}(x).$$

Consequently, $A\Theta_{s;\,ja}^{(m)\times}(x) = \lambda\Theta_{s;\,ja}^{(m)\times}(x)$, where $\lambda = \mathcal{A}(-\widehat{p^{j}}s)$.

Conversely, if $A\Theta_{s;\,ja}^{(m)\times}(x) = \lambda\Theta_{s;\,ja}^{(m)\times}(x)$, $\lambda \in \mathbb{C}$, taking the Fourier transform of both left- and right-hand sides of this identity and using (4.1), (3.27), (4.7), we have

$$\big(\mathcal{A}(\xi) - \lambda\big)\chi_p\big(\widehat{p^{-j}}a \cdot \xi\big)\Omega\big(|s + \widehat{p^{-j}}\xi|_p\big) = 0, \quad \xi \in \mathbb{Q}_p^n. \tag{4.8}$$

Now, if $s + \widehat{p^{-j}}\xi = \eta$, $\eta \in \mathbb{Z}_p^n$, then $\xi = \widehat{p^{j}}(-s + \eta)$. Since $\chi_p\big(\widehat{p^{-j}}a \cdot \xi\big) \neq 0$, $\Omega\big(|s+\widehat{p^{-j}}\xi|_p\big) \neq 0$, it follows from (4.8) that $\lambda = \mathcal{A}\big(\widehat{p^{j}}(-s+\eta)\big)$ for any $\eta \in \mathbb{Z}_p^n$. Thus $\lambda = \mathcal{A}(-\widehat{p^{j}}s)$, and, consequently, (4.6) holds.

The proof of the theorem is complete. $\square$

Corollary 4.1. *Let A be a pseudo-differential operator* (4.1) *with the symbol $\mathcal{A}(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$. Then the n-dimensional wavelet function* (3.26)

$$\Theta_{k;ja}^{\times}(x) = p^{-|j|/2}\chi_p\big(p^{-1}k \cdot (\widehat{p^{j}}x - a)\big)\Omega\big(|\widehat{p^{j}}x - a|_p\big), \quad x \in \mathbb{Q}_p^n,$$

is an eigenfunction of A if and only if

$$\mathcal{A}\big(\widehat{p^{j}}(-p^{-1}k + \eta)\big) = \mathcal{A}\big(-\widehat{p^{j-I}}k\big), \qquad \forall\, \eta \in \mathbb{Z}_p^n,$$

where $k \in J_p^n$, $j \in \mathbb{Z}^n$, $a \in I_p^n$, $I = (1, \dots, 1)$. The corresponding eigenvalue is $\lambda = \mathcal{A}\big(-\widehat{p^{j-I}}j\big)$, i.e.,

$$A\Theta_{k;ja}^{\times}(x) = \mathcal{A}(-\widehat{p^{j-I}}k)\Theta_{k;ja}^{\times}(x).$$

Representation (3.17) and Theorem 4.1 imply the following statement.

Theorem 4.2. *Let A be a pseudo-differential operator* (4.1) *with the symbol $\mathcal{A}(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$. Then the n-dimensional wavelet function* (3.29)

$$\Psi_{s;\,ja}^{(m)[\nu]\times}(x) = p^{-|j|/2}\psi_s^{(m_1)[\nu]}(p^{j_1}x_1 - a_1)\cdots\psi_s^{(m_n)[\nu]}(p^{j_n}x_n - a_n), \quad x \in \mathbb{Q}_p^n,$$

is an eigenfunction of A if and only if condition (4.6) *holds, where $\psi_{s_{j_k}}^{(m_{j_k})[\nu]}$ is defined by* (3.17), (3.18), *$j = (j_1, \dots, j_n) \in \mathbb{Z}^n$; $|j| = j_1 + \cdots + j_n$; $a = (a_1, \dots, a_n) \in I_p^n$; $s = (s_1, \dots, s_n) \in J_{p;m}^n$; $m = (m_1, \dots, m_n)$, $m_k \geq 1$ is a fixed positive integer, $k = 1, 2, \dots, n$; $\nu = 1, 2, \dots$. The corresponding eigenvalue is $\lambda = \mathcal{A}(-\widehat{p^{j}}s)$, i.e.,*

$$A\Psi_{s;\,ja}^{(m)[\nu]\times}(x) = \mathcal{A}(-\widehat{p^{j}}s)\Psi_{s;\,ja}^{(m)[\nu]\times}(x).$$

4.4. **p-Adic wavelets as eigenfunctions of the Taibleson fractional operator.** As mentioned above, the Taibleson fractional operator D^{α} has the symbol $\mathcal{A}(\xi) = |\xi|_p^{\alpha}$. The symbol $\mathcal{A}(\xi) = |\xi|_p^{\alpha}$ satisfies the condition (4.6):

$$\mathcal{A}\big(\widehat{p^{j}}(-s + \eta)\big) = |\widehat{p^{j}}(-s + \eta)|_p^{\alpha} = \Big(\max_{1 \leq r \leq n}\big(p^{-j_r}| - s_r|_p\big)\Big)^{\alpha}$$

$$= \mathcal{A}\big(-\widehat{p^{j}}s\big) = p^{\alpha \max_{1 \leq r \leq n}\{m_r - j_r\}}$$

for all $\eta \in \mathbb{Z}_p^n$. Consequently, according to Theorem 4.1, we have

Corollary 4.2. *The n-dimensional non-Haar p-adic wavelet* (3.25) *is an eigenfunction of the Taibleson fractional operator* (4.4)*:*

$$D^{\alpha}\Theta_{s;\,ja}^{(m)\times}(x) = p^{\alpha \max_{1\le r\le n}\{m_r-j_r\}}\Theta_{s;\,ja}^{(m)\times}(x), \quad \alpha \in \mathbb{C}, \qquad x \in \mathbb{Q}_p^n,$$

$s \in J_{p;m}^n$, $j \in \mathbb{Z}^n$, $a \in I_p^n$.

In particular, in view of Corollary 4.1, we have

Corollary 4.3. *The n-dimensional p-adic wavelet* (3.26) *is an eigenfunction of the Taibleson fractional operator* (4.4)*:*

$$D^{\alpha}\Theta_{k;\,ja}^{\times}(x) = p^{\alpha(1-\min_{1\le r\le n} j_r)}\Theta_{k;\,ja}^{\times}(x), \quad \alpha \in \mathbb{C}, \quad x \in \mathbb{Q}_p^n,$$

$k \in J_p^n$, $j \in \mathbb{Z}^n$, $a \in I_p^n$.

Corollary 4.4. *The n-dimensional p-adic wavelet* (3.29) *is an eigenfunction of the Taibleson fractional operator* (4.4)*:*

$$D^{\alpha}\Psi_{s;\,ja}^{(m)[\nu]\times}(x) = p^{\alpha \max_{1\le r\le n}\{m_r-j_r\}}\Psi_{s;\,ja}^{(m)[\nu]\times}(x), \quad \alpha \in \mathbb{C}, \quad x \in \mathbb{Q}_p^n,$$

$s \in J_{p;m}^n$, $j \in \mathbb{Z}^n$, $a \in I_p^n$, $\nu = 1, 2, \dots$.

5. Application of p-adic wavelets to evolutionary pseudo-differential equations

5.1. **Linear equations.** **(a)** Let us consider the Cauchy problem for the *linear evolutionary pseudo-differential equation*

$$\left\{\begin{array}{rcll} \frac{\partial u(x,t)}{\partial t} + A_x u(x,t) & = & 0, & \text{in} \quad \mathbb{Q}_p^n \times (0,\ \infty), \\ u(x,t) & = & u^0(x), & \text{in} \quad \mathbb{Q}_p^n \times \{t=0\}, \end{array}\right. \tag{5.1}$$

where $t \in \mathbb{R}$, $u^0 \in \Phi'(\mathbb{Q}_p^n)$ and

$$A_x u(x,t) = F^{-1}\big[\mathcal{A}(\xi)\, F[u(\cdot,t)](\xi)\big](x) \tag{5.2}$$

is a pseudo-differential operator (4.1) (with respect to x) with symbols $\mathcal{A}(\xi) \in \mathcal{E}(\mathbb{Q}_p^n \setminus \{0\})$, $u(x,t)$ is the desired distribution such that $u(x,t) \in \Phi'(\mathbb{Q}_p^n)$ for any $t \ge 0$.

In particular, we will consider the Cauchy problem

$$\left\{\begin{array}{rcll} \frac{\partial u(x,t)}{\partial t} + D_x^{\alpha} u(x,t) & = & 0, & \text{in} \quad \mathbb{Q}_p^n \times (0,\ \infty), \\ u(x,t) & = & u^0(x), & \text{in} \quad \mathbb{Q}_p^n \times \{t=0\}, \end{array}\right. \tag{5.3}$$

where $D_x^{\alpha} u(x,t) = F^{-1}\big[|\xi|_p^{\alpha}\, F[u(\cdot,t)](\xi)\big](x)$ is the Taibleson fractional operator (4.3) with respect to x, $\alpha \in \mathbb{C}$.

Theorem 5.1. *The Cauchy problem* (5.1) *has a unique solution*

$$u(x,t) = F^{-1}\big[F[u^0(\cdot)](\xi) e^{-\mathcal{A}(\xi)\,t}\big](x). \tag{5.4}$$

Proof. Since $u(x,t)$ is a distribution such that $u(x,t) \in \Phi'(\mathbb{Q}_p^n)$ for any $t \geq 0$, the relation (5.1) is well-defined. Applying the Fourier transform to (5.1), we obtain the following equation

$$\frac{\partial F[u(\cdot,t)](\xi)}{\partial t} + \mathcal{A}(\xi)\, F[u(\cdot,t)](\xi) = 0.$$

Solving this equation, we obtain

$$F[u(x,t)](\xi) = F[u(x,0)](\xi) e^{-\mathcal{A}(\xi)\, t}.$$

This implies (5.4). □

Theorem 5.2. *Let a pseudo-differential operator A_x in* (5.1) *be such that its symbol $\mathcal{A}(\xi)$ satisfies the condition* (4.6)*:*

$$\mathcal{A}\big(\widehat{p^j}(-s+\eta)\big) = \mathcal{A}\big(-\widehat{p^j}s\big), \qquad \forall\, \eta \in \mathbb{Z}_p^n,$$

for any $j \in \mathbb{Z}^n$, $s \in J_{p;m}^n$. Then the Cauchy problem (5.1) *has a unique solution*

$$u(x,t) = \sum_{s \in J_{p;m}^n, j \in \mathbb{Z}^n, a \in I_p^n} \big\langle u^0(x), \Theta_{s;\,ja}^{(m)\times} \big\rangle e^{-\mathcal{A}(-\widehat{p^j}s)t} \Theta_{s;\,ja}^{(m)\times}(x), \tag{5.5}$$

for $t \geq 0$, where $\Theta_{s;\,ja}^{(m)\times}(x)$ are n-dimensional p-adic wavelets (3.25)*.*

Proof. According to the formula (3.32) from Proposition 3.1, we will seek a solution of the Cauchy problem (5.1) in the form of an infinite sum

$$u(x,t) = \sum_{s \in J_{p;m}^n, j \in \mathbb{Z}^n, a \in I_p^n} \Lambda_{s;\,j,a}(t) \Theta_{s;\,ja}^{(m)\times}(x), \tag{5.6}$$

where $\Lambda_{s;\,j,a}(t)$ are the desired functions, $s \in J_{p;m}^n$, $j \in \mathbb{Z}^n$, $a \in I_p^n$.

Substituting (5.6) into equation (5.1), in view of of Theorem 4.1, we obtain

$$\sum_{s \in J_{p;m}^n, j \in \mathbb{Z}^n, a \in I_p^n} \Big(\frac{d\Lambda_{s;\,j,a}(t)}{dt} + \mathcal{A}(-\widehat{p^j}s)\Lambda_{s;\,j,a}(t)\Big)\Theta_{s;\,ja}^{(m)\times}(x) = 0.$$

The last equation is understood in the weak sense, i.e.,

$$\sum_{s \in J_{p;m}^n, j \in \mathbb{Z}^n, a \in I_p^n} \Big\langle \Big(\frac{d\Lambda_{s;\,j,a}(t)}{dt} + \mathcal{A}(-\widehat{p^j}s)\Lambda_{s;\,j,a}(t)\Big)\Theta_{s;\,ja}^{(m)\times}(x), \phi(x)\Big\rangle = 0, \tag{5.7}$$

for all $\phi \in \Phi(\mathbb{Q}_p^n)$. Since according to Lemma 3.1, any test function $\phi \in \Phi(\mathbb{Q}_p^n)$ is represented in the form of a *finite* sum (3.30), the equality (5.7) implies that

$$\frac{d\Lambda_{s;\,j,a}(t)}{dt} + \mathcal{A}(-\widehat{p^j}s)\Lambda_{s;\,j,a}(t) = 0, \quad \forall\, s \in J_{p;m}^n,\ j \in \mathbb{Z}^n,\ a \in I_p^n,$$

for all $t \geq 0$. Solving this differential equation, we obtain

$$\Lambda_{s;\,j,a}(t) = \Lambda_{s;\,j,a}(0) e^{-\mathcal{A}(-\widehat{p^j}s)t}, \quad s \in J_{p;m}^n, \quad j \in \mathbb{Z}^n, \quad a \in I_p^n. \tag{5.8}$$

By substituting (5.8) into (5.6) we find a solution of the Cauchy problem (5.1) in the form

$$u(x,t) = \sum_{s\in J^n_{p;m}, j\in\mathbb{Z}^n, a\in I^n_p} \Lambda_{s;\,j,a}(0) e^{-\mathcal{A}(-\widehat{p^j s})t} \Theta^{(m)\times}_{s;\,ja}(x). \tag{5.9}$$

Setting $t=0$, we find

$$u^0(x) = \sum_{s\in J^n_{p;m}, j\in\mathbb{Z}^n, a\in I^n_p} \Lambda_{s;\,j,a}(0) \Theta^{(m)\times}_{s;\,ja}(x),$$

where $u^0 \in \Phi'(\mathbb{Q}^n_p)$ and according to (3.32), the coefficients $\Lambda_{s;\,j,a}(0)$ are uniquely determined by (3.33) as

$$\Lambda_{s;\,j,a}(0) = \big\langle u^0(x), \Theta^{(m)\times}_{s;\,ja}\big\rangle, \quad s\in J^n_{p;m}, \quad j\in\mathbb{Z}^n, \quad a\in I^n_p. \tag{5.10}$$

The relations (5.9), (5.10) imply (5.5). In view of (3.34), the sum (5.5) is finite on any test function from the Lizorkin space $\Phi(\mathbb{Q}^n_p)$.

The theorem is thus proved. □

Theorem 5.2 and Corollary 4.2 imply the following assertion.

Corollary 5.1. *The Cauchy problem* (5.3) *has a unique solution*

$$u(x,t) = \sum_{s\in J^n_{p;m}, j\in\mathbb{Z}^n, a\in I^n_p} \big\langle u^0(x), \Theta^{(m)\times}_{s;\,ja}\big\rangle e^{-p^{\alpha \max_{1\le r\le n}\{m_r-j_r\}}t} \Theta^{(m)\times}_{s;\,ja}(x), \tag{5.11}$$

for $t\ge 0$, where $\Theta^{(m)\times}_{s;\,ja}(x)$ are n-dimensional p-adic wavelets (3.25).

Solutions of the Cauchy problems (5.1) and (5.3) describe the diffusion processes in the space $\mathbb{Q}^n_p$.

If $\mathcal{A}(-\widehat{p^j s}) > 0$, according to (5.5), $u(x,t)\to 0$, as $t\to\infty$. In particular, this fact holds for the solution of the Cauchy problem (5.3).

Example 5.1. Consider the one-dimensional Cauchy problem (5.3) for the initial data

$$u^0(x) = \Omega(|x|_p) = \begin{cases} 1, & |x|_p \le 1, \\ 0, & |x|_p > 1. \end{cases}$$

Substituting (3.4), (3.11), (3.12) into (5.11), we obtain a solution of this Cauchy problem:

$$u(x,t) = \sum_{s\in J_{p;m}} \sum_{j=m}^{\infty} p^{-j} e^{-p^{\alpha(m-j)}t} \chi_p\big(sp^j x\big)\Omega\big(|p^j x|_p\big).$$

(b) Now we consider the Cauchy problem

$$\begin{cases} i\frac{\partial u(x,t)}{\partial t} - A_x u(x,t) &= 0, \qquad \text{in} \quad \mathbb{Q}^n_p\times(0,\ \infty), \\ \qquad\qquad u(x,t) &= u^0(x), \quad \text{in} \quad \mathbb{Q}^n_p\times\{t=0\}, \end{cases} \tag{5.12}$$

where $u^0 \in \Phi'(\mathbb{Q}_p^n)$ and a pseudo-differential operator operator A_x is given by (5.2). In particular, we have the Cauchy problem

$$\text{(5.13)} \qquad \begin{cases} i\frac{\partial u(x,t)}{\partial t} - D_x^{\alpha} u(x,t) = 0, & \text{in} \quad \mathbb{Q}_p^n \times (0,\ \infty), \\ \qquad\qquad u(x,t) = u^0(x), & \text{in} \quad \mathbb{Q}_p^n \times \{t=0\}, \end{cases}$$

where D_x^{α} is the Taibleson fractional operator (4.3) with respect to x, $\alpha \in \mathbb{C}$.

Using the above results, one can construct a solution of the Cauchy problems (5.12) and (5.13).

Theorem 5.3. *Let a pseudo-differential operator A_x in* (5.12) *be such that its symbol $\mathcal{A}(\xi)$ satisfies the condition* (4.6)*:*

$$\mathcal{A}\big(\widehat{p^j}(-s+\eta)\big) = \mathcal{A}\big(-\widehat{p^j}s\big), \qquad \forall\, \eta \in \mathbb{Z}_p^n,$$

for any $j \in \mathbb{Z}^n$, $s \in J_{p;m}^n$. Then the Cauchy problem (5.12) *has a unique solution*

$$\text{(5.14)} \qquad u(x,t) = \sum_{s \in J_{p;m}^n, j \in \mathbb{Z}^n, a \in I_p^n} \big\langle u^0(x), \Theta_{s;\,ja}^{(m)\times} \big\rangle e^{-i\mathcal{A}(-\widehat{p^j}s)t} \Theta_{s;\,ja}^{(m)\times}(x),$$

for $t \ge 0$, where $\Theta_{s;\,ja}^{(m)\times}(x)$ are n-dimensional p-adic wavelets (3.25)*.*

Corollary 5.2. *The Cauchy problem* (5.13) *has a unique solution*

$$\text{(5.15)} \quad u(x,t) = \sum_{s \in J_{p;m}^n, j \in \mathbb{Z}^n, a \in I_p^n} \big\langle u^0(x), \Theta_{s;\,ja}^{(m)\times} \big\rangle e^{-ip^{\alpha \max_{1 \le r \le n}\{m_r - j_r\}}t} \Theta_{s;\,ja}^{(m)\times}(x),$$

for $t \ge 0$, where $\Theta_{s;\,ja}^{(m)\times}(x)$ are n-dimensional p-adic wavelets (3.25)*.*

5.2. **Semi-linear equations.** Consider the Cauchy problem for the semi-linear pseudo-differential equation:

$$\text{(5.16)} \quad \begin{cases} \frac{\partial u(x,t)}{\partial t} + A_x u(x,t) + u(x,t)|u(x,t)|^{2m} = 0, & \text{in} \quad \mathbb{Q}_p^n \times (0,\ \infty), \\ \qquad\qquad\qquad u(x,t) = u^0(x), & \text{in} \quad \mathbb{Q}_p^n \times \{t=0\}, \end{cases}$$

where pseudo-differential operator A_x is given by (5.2), $m \in \mathbb{N}$, $u(x,t)$ is the desired distribution such that $u(x,t) \in \Phi'(\mathbb{Q}_p^n)$ for any $t \ge 0$.

According to Proposition 3.1, a distribution $u(x,t)$ can be realized as an infinite sum of the form

$$\text{(5.17)} \qquad u(x,t) = \sum_{s \in J_{p;m}^n, j \in \mathbb{Z}^n, a \in I_p^n} \Lambda_{s;\,j,a}(t) \Theta_{s;\,ja}^{(m)\times}(x),$$

where $\Lambda_{s;\,j,a}(t)$ are the desired functions, $\Theta_{s;\,ja}^{(m)\times}(x)$ are elements of the wavelet basis (3.25). We will solve the Cauchy problem in a particular class of distributions $u(x,t)$ such that in representation (5.17)

$$\text{(5.18)} \qquad \widehat{p^{j'-j}}a - a' \notin \mathbb{Z}_p^n, \quad \text{if} \quad j_k < j'_k, \quad k = 1, \dots, n.$$

In view of (3.6), in this case all sets $\{x \in \mathbb{Q}_p^n : |\widehat{p^j}x - a|_p \le 1\}$, $\{x \in \mathbb{Q}_p^n : |\widehat{p^{j'}}x - a'|_p \le 1\}$ are disjoint.

Theorem 5.4. *Let a pseudo-differential operator A_x in* (5.16) *be such that its symbol $\mathcal{A}(\xi)$ satisfies the condition* (4.6)*:*

$$\mathcal{A}\big(\widehat{p^j}(-s+\eta)\big)=\mathcal{A}\big(-\widehat{p^j}s\big), \qquad \forall\,\eta\in\mathbb{Z}_p^n,$$

for any $j\in\mathbb{Z}^n$, $s\in J_{p;m}^n$. Then in the above-mentioned class of distributions (5.17), (5.18) *the Cauchy problem* (5.16) *has a unique solution*

$$u(x,t)$$

$$\sum_{s\in J_{p;m}^n,j\in\mathbb{Z}^n,a\in I_p^n}\frac{\langle u^0(x),\Theta_{s;\,ja}^{(m)\times}\rangle\big(\mathcal{A}(-\widehat{p^j}s)\big)^{1/2m}e^{-\mathcal{A}(-\widehat{p^j}s)t}\,\Theta_{s;\,ja}^{(m)\times}(x)}{\big(\mathcal{A}(-\widehat{p^j}s)+\langle u^0(x),\Theta_{s;\,ja}^{(m)\times}\rangle^{2m}p^{-m|j|}\big(1-e^{-2m\mathcal{A}(-\widehat{p^j}s)t}\big)\big)^{1/2m}} \tag{5.19}$$

for $t\ge 0$, where $\Theta_{s;\,ja}^{(m)\times}(x)$ are n-dimensional p-adic wavelets (3.25)*. Moreover, this formula is applicable for the case $\mathcal{A}\equiv 0$.*

Proof. Since $|\chi_p\big(p^{-1}k\cdot(p^jx-a)\big)|=1$, taking into account formulas (5.17), (5.18), (3.25), we obtain

$$|u(x,t)|^2=\sum_{j\in\mathbb{Z},k\in J_{p\,0}^n,a\in I_p^n}\Lambda_{s;\,j,a}^2(t)p^{-|j|}\Omega\big(|\widehat{p^j}x-a|_p\big)$$

and

$$u(x,t)|u(x,t)|^{2m}=\sum_{j\in\mathbb{Z},k\in J_{p\,0}^n,a\in I_p^n}\Lambda_{s;\,j,a}^{2m+1}(t)p^{-m|j|}\Theta_{s;\,ja}^{(m)\times}(x), \tag{5.20}$$

where the indexes in the above sums satisfy the condition (5.18).

Substituting (5.20) and (5.17) into (5.16), in view of of Theorem 4.1, we find that

$$\sum_{j\in\mathbb{Z},k\in J_{p\,0}^n,a\in I_p^n}\Big(\frac{d\Lambda_{s;\,j,a}(t)}{dt}+\mathcal{A}\big(-\widehat{p^j}s\big)\Lambda_{s;\,j,a}(t)$$

$$+\,p^{-m|j|}\Lambda_{s;\,j,a}^{2m+1}(t)\Big)\Theta_{s;\,ja}^{(m)\times}(x)=0, \tag{5.21}$$

where the last equation is understood in the weak sense. Since, according to Lemma 3.1, any test function $\phi\in\Phi(\mathbb{Q}_p^n)$ is represented in the form of a *finite* sum (3.30), the equality (5.21) implies that

$$\frac{d\Lambda_{s;\,j,a}(t)}{dt}+\mathcal{A}\big(-\widehat{p^j}s\big)\Lambda_{s;\,j,a}(t)$$

$$+\,p^{-m|j|}\Lambda_{s;\,j,a}^{2m+1}(t)=0, \quad \forall\,s\in J_{p;m}^n,\ j\in\mathbb{Z}^n,\ a\in I_p^n, \tag{5.22}$$

for all $t\ge 0$. Integrating (5.22), we obtain

$$\frac{\Lambda_{s;\,j,a}^{2m}(t)}{\mathcal{A}(-\widehat{p^j}s)+p^{-m|j|}\Lambda_{s;\,j,a}^{2m}(t)}=E_{s;\,j,a}e^{-2m\mathcal{A}(-\widehat{p^j}s)t},$$

i.e.,

$$\Lambda_{s;\,j,a}(t) = \frac{E_{s;\,j,a}^{1/2m}\big(\mathcal{A}(-\widehat{p^{j}}s)\big)^{1/2m}e^{-\mathcal{A}(-\widehat{p^{j}}s)t}}{\big(1-E_{s;\,j,a}p^{-m|j|}e^{-2m\mathcal{A}(-\widehat{p^{j}}s)t}\big)^{1/2m}}, \tag{5.23}$$

where $E_{s;\,j,a}$ is a constant, $s\in J_{p;m}^{n}$, $j\in\mathbb{Z}^{n}$, $a\in I_{p}^{n}$. Substituting (5.23) into (5.17), we find a solution of the problem (5.16)

$$u(x,t) = \sum_{s\in J_{p;m}^{n},j\in\mathbb{Z}^{n},a\in I_{p}^{n}} \frac{E_{s;\,j,a}^{1/2m}\big(\mathcal{A}(-\widehat{p^{j}}s)\big)^{1/2m}e^{-\mathcal{A}(-\widehat{p^{j}}s)t}}{\big(1-E_{s;\,j,a}p^{-m|j|}e^{-2m\mathcal{A}(-\widehat{p^{j}}s)t}\big)^{1/2m}}\Theta_{s;\,ja}^{(m)\times}(x), \tag{5.24}$$

$x\in\mathbb{Q}_{p}^{n}$, $t\ge 0$. Setting in (5.24) $t=0$, we obtain that

$$u^{0}(x) = \sum_{s\in J_{p;m}^{n},j\in\mathbb{Z}^{n},a\in I_{p}^{n}} \left(\frac{E_{s;\,j,a}\mathcal{A}(-\widehat{p^{j}}s)}{1-E_{s;\,j,a}p^{-m|j|}}\right)^{1/2m}\Theta_{s;\,ja}^{(m)\times}(x),$$

where $u^{0}\in\Phi'(\mathbb{Q}_{p}^{n})$. Hence, according to (3.32), the coefficients $E_{s;\,j,a}$ are uniquely determined by (3.33) as

$$\left(\frac{E_{s;\,j,a}\mathcal{A}(-\widehat{p^{j}}s)}{1-E_{s;\,j,a}p^{-m|j|}}\right)^{1/2m} = \big\langle u^{0}(x),\Theta_{s;\,ja}^{(m)\times}\big\rangle.$$

The last equation implies that

$$E_{s;\,j,a} = \frac{\langle u^{0}(x),\Theta_{s;\,ja}^{(m)\times}\rangle^{2m}}{\mathcal{A}(-\widehat{p^{j}}s)+p^{-m|j|}\langle u^{0}(x),\Theta_{s;\,ja}^{(m)\times}\rangle^{2m}}.$$

Substituting $E_{s;\,j,a}$ into (5.24), we obtain (5.19). In view of (3.34), the sum (5.19) is finite on any test function from the Lizorkin space $\Phi(\mathbb{Q}_{p}^{n})$.

Now by passing to the limit as $\mathcal{A}\to 0$ in formula (5.19), one can easily see that this formula (5.19) is applicable for the case $\mathcal{A}\equiv 0$.

The theorem is thus proved. □

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